THE LINEARIZED 2D INVISCID SHALLOW WATER EQUATIONS IN A RECTANGLE: BOUNDARY CONDITIONS AND WELL-POSEDNESS

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ABSTRACT. We consider the linearized 2D inviscid shallow water equations in a rectangle. A set of boundary conditions is proposed which make these equations well-posed. Several different cases occur depending on the relative values of the reference velocities (u_0, v_0) and reference height ϕ_0 (sub- or super-critical flow at each part of the boundary).

1. INTRODUCTION

Motivated by the study of the inviscid linearized primitive equation (as a step towards the study of the full nonlinear equations), we consider in this article the linearized fully inviscid shallow water equations in a rectangle. To the best of our knowledge, the well-posedness of the linearized inviscid shallow water equations in a rectangle has not yet been addressed. In particular, the existence of corners in the geometrical domain is a subject of mathematical concern since the seminal works [Osh73, Osh74] which show the possible occurrence of major singularities in the corners for certain choices of the boundary conditions. The boundary conditions that we choose do not lead to singularities and allow us to develop results of existence and uniqueness of a solution. As explained below, in the rationale of the Local Area Models (LAMs) to which this work relates (see e.g. [RTT08a, RTT08b, WPT97]), we are not looking for all the possible boundary conditions which make the equations well-posed, but only certain sets of such boundary conditions, any set in fact.

In this article, the well-posedness of the linearized inviscid shallow water equations is classically conducted using the linear semigroup theory. We write these equations as an abstract evolution equation in a Hilbert space:

$$\frac{dU}{dt} + AU = F,$$

with initial data $U(0) = U_0$ given, which means that we prescribe the boundary conditions at $t = 0$ (see below). In this approach, the main difficulty is to show that $\langle AU, U\rangle \geq 0$ for all U in the domain $\mathcal{D}(A)$ of A (and similarly $\langle A^*U, U\rangle \geq 0$, $\forall U \in \mathcal{D}(A^*)$), where A is the abstract associated operator acting in $L^2(\Omega)^3$. This result is achieved using a suitable approximation technique. Five different cases occur depending on whether the linearized base flow with velocities (u_0, v_0) and height ϕ_0 is sub- or super-critical on each side of the rectangle ($u_0^2, v_0^2 >$ or $< g\phi_0$). The fifth case corresponding to $u_0^2 + v_0^2 < g\phi_0$ is fully subcritical ($u_0^2, v_0^2 < g\phi_0$); furthermore the time independent part of the shallow water equations is partly elliptic in this case, but of course the full shallow water system remains hyperbolic. Boundary value problems of the Dirichlet or Neumann type appear in this case.

Date: September 17, 2012.

Of course the presence of the corners also raises difficulties for these elliptic problems, but the desired regularity of the solutions of the corresponding Dirichlet-Neumann problem is achieved using results from [Gri85].

The study of the Local Area Models (LAMs) in the atmosphere and oceans sciences, leads to initial and boundary value problems for the inviscid primitive equations in these domains, and we know since [OS78] (see also [TT03]) that these problems are complicated leading to nonlocal boundary value problems. As explained in e.g. [WPT97] the choice of the boundary condition is important for the numerical simulations, as one wishes boundary conditions leading to well-posed problems (to avoid numerical explosion) and boundary conditions that are transparent, letting the waves move freely inside and outside the domain. In the perspective of a LAM imbedded in a large domain, any such set of conditions is acceptable.

The linearized shallow water equations that we consider read

$$\begin{cases} u_t + u_0 u_x + v_0 u_y + g\phi_x - fv = 0, \\ v_t + u_0 v_x + v_0 v_y + g\phi_y + fu = 0, \\ \phi_t + u_0\phi_x + v_0\phi_y + \phi_0(u_x + v_y) = 0; \end{cases} \tag{1.1}$$

here $U = (u, v, \phi)^t, (x, y) \in \Omega := (0, L_1) \times (0, L_2)$, u_0, v_0, ϕ_0 are positive constants, and g is the gravitational acceleration, f is the Coriolis parameter. As explained in e.g. [OS78, TT03, RTT08b], the inviscid primitive equations in a cube can be expanded in vertical modes expansion producing a set of coupled bi-dimensional equations, similar to coupled shallow water equations. For one linearized mode, the system that we obtain is very similar to (1.1) as we show at the end of this article. With this respect the results that we obtain for the one-mode linearized primitive equations generalize in part the results from [RTT08b] in which we assumed $v_0 = 0$; the full coupled system as considered in [RTT08b] will be studied elsewhere.

As we know the literature on the shallow water equations is very vast, both on the theoretical and computational aspects, considering the viscous equations or the partly or totally inviscid equations and considering that the height is either always strictly positive or that it can vanish. See e.g. [ABBKP, AB03, ABPS, SLTT, BPSTT] on the computational side and see e.g. [Ore95, PT10, PT11, HPT11, RTT08a] on the theoretical side. But as we said we did not find this problem studied in the literature. The general hyperbolic equations in a domain with corners are also considered in [Tan78, KT80], but the boundary conditions they imposed is different from ours, and the space they used is not the usual L^2 space which we use in this article.

This article is organized as follows. After this introductory section, we describe in Section 2 the boundary conditions which one can associate to the time independent part $\mathcal{A}$ of the shallow water equations (dropping also the Coriolis terms). We also present a first useful trace theorem. Then in Section 3 we derive various density theorems, density of certain smooth functions in certain function spaces. Section 4 is devoted to the fully supercritical case when u_0^2 and v_0^2 are both larger than $g\phi_0$; Section 5 is devoted to the mixed cases when $u_0^2 > g\phi_0$ and $v_0^2 < g\phi_0$ or vice versa; Section 6 tackles the fully hyperbolic subcritical case corresponding to $u_0^2 < g\phi_0$ and $v_0^2 < g\phi_0$ whereas $u_0^2 + v_0^2 > g\phi_0$. Section 7 is a technical section giving a regularity result for an associated elliptic problem. It prepares to Section 8 in which we deal with the mixed subcritical case, where appears a partly elliptic and

partly hyperbolic operator $\mathcal{A}$; this case corresponds to $u_0^2 + v_0^2 < g\phi_0$, so that $u_0^2 < g\phi_0$ and $v_0^2 < g\phi_0$. In Section 9 we state the general existence and uniqueness theorem proved by application of the Hille-Phillips-Yoshida theorem, which covers all cases. We finally make the connection with the equations for one mode of the linearized primitive equations and interpret our earlier result for that case.

We conclude this introduction with a last remark which will be slightly reworded in the final version of this article. This work is at the interface between hyperbolic equations and parabolic-elliptic equations; the equations are hyperbolic, the methods used are rather those of parabolic and elliptic equations. We believe that the results presented here are new; if they are known, they were at least not known by us as we said nor by those working in geophysical fluid mechanics who need them most [WPT97]**. The authors are not specialists of hyperbolic equations and they may have missed some relevant references. They will gratefully receive any suggestion from the Editors or from the Referees on the bibliography or else. In some sense, this article is a continuation of** [RTT08b] **published in the Journal de Mathématiques Pures et Appliquées which dealt with the case where $v_0 = 0$. However, the methods here are different. Furthermore beside the four cases that one can surmise by comparison with** [RTT08b]**, there is a fifth unexpected case which is the object of a specific study in Section 8. Finally in** [RTT08b]**, we related the system to the primitive equations of the atmosphere and the oceans. Here we rather relate it to the shallow water equations, but the systems are essentially the same, the relation between these two systems is clarified in Remark 9.2.**

2. Boundary conditions

In order to derive the boundary conditions for (1.1), we introduce the time independent 2D shallow water equations operator:

$$\mathcal{A}U := \mathcal{A}\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} u_0 u_x + v_0 u_y + g\phi_x \\ u_0 v_x + v_0 v_y + g\phi_y \\ u_0\phi_x + v_0\phi_y + \phi_0(u_x + v_y) \end{pmatrix}; \tag{2.1}$$

we set

$$\mathcal{E}_1 = \begin{pmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ \phi_0 & 0 & u_0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & \phi_0 & v_0 \end{pmatrix};$$

and we aim to determine boundary conditions which are suitable for the system

$$\mathcal{A}U = \mathcal{E}_1 U_x + \mathcal{E}_2 U_y = F. \tag{2.2}$$

In the following, we assume that $u_0 > 0, v_0 > 0$. The case where u_0 and, or v_0 are negative can be treated in a similar manner; we do not study the non-generic case where one or both of u_0, v_0 vanish. By its physical meaning (height), $\phi_0 \geq 0$ and we do not study the case where $\phi_0 = 0$. Similary, we only study the generic cases where $u_0^2 \neq g\phi_0$, $v_0^2 \neq g\phi_0$ and $u_0^2 + v_0^2 \neq g\phi_0$. As we will see below, we have five cases to study, for four of them equations (2.2) are fully hyperbolic and for the fifth one the equations are partly hyperbolic and partly elliptic.

We first consider the situation when $u_0^2+v_0^2-g\phi_0$ is positive, which produces four fully hyperbolic cases, and we call them the supercritical case, the mixed case (two sub-cases), and the fully hyperbolic subcritical case. Let us set $\kappa_0=\sqrt{g(u_0^2+v_0^2-g\phi_0)/\phi_0}$.

2.1. **The supercritical case.** The supercritical (fully supersonic) case corresponds to the case where:

$$u_0^2>g\phi_0,\ \ v_0^2>g\phi_0. \tag{2.3}$$

We compute the eigenvalues of $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\begin{aligned}&\text{Eigenvalues of } \mathcal{E}_1\text{: } u_0-\sqrt{g\phi_0}, u_0, u_0+\sqrt{g\phi_0},\\ &\text{Eigenvalues of } \mathcal{E}_2\text{: } v_0-\sqrt{g\phi_0}, v_0, v_0+\sqrt{g\phi_0}.\end{aligned} \tag{2.4}$$

With assumption (2.3), we see that all the eigenvalues of $\mathcal{E}_1$ and $\mathcal{E}_2$ are positive. Thus, it is natural to treat either the x- or y-direction as the time-like direction. Let us choose the y-direction, which means that we first need to specify the boundary conditions at $y=0$ (time-like initial conditions). Multiplying both sides of (2.2) by $\mathcal{E}_2^{-1}$ gives

$$U_y+\mathcal{E}_2^{-1}\mathcal{E}_1U_x=\mathcal{E}_2^{-1}F. \tag{2.5}$$

Directly or on using Matlab, we compute

$$\mathcal{E}_2^{-1}\mathcal{E}_1=\frac{1}{v_0^2-g\phi_0}\begin{pmatrix}\frac{u_0}{v_0}(v_0^2-g\phi_0) & 0 & \frac{g}{v_0}(v_0^2-g\phi_0)\\ -g\phi_0 & u_0v_0 & -gu_0\\ v_0\phi_0 & -u_0\phi_0 & u_0v_0\end{pmatrix}, \tag{2.6}$$

and write

$$P^{-1}\cdot\mathcal{E}_2^{-1}\mathcal{E}_1\cdot P=diag(\lambda_1,\lambda_2,\lambda_3), \tag{2.7}$$

where P has a complicated expression, whereas

$$P^{-1}=\begin{pmatrix}\frac{v_0}{2\kappa_0} & -\frac{u_0}{2\kappa_0} & \frac{1}{2}\\ -\frac{v_0}{2\kappa_0} & \frac{u_0}{2\kappa_0} & \frac{1}{2}\\ \frac{u_0v_0}{u_0^2+v_0^2} & \frac{v_0^2}{u_0^2+v_0^2} & \frac{gv_0}{u_0^2+v_0^2}\end{pmatrix},$$

and

$$\lambda_1=\frac{u_0v_0+\phi_0\kappa_0}{v_0^2-g\phi_0},\ \ \lambda_2=\frac{u_0v_0-\phi_0\kappa_0}{v_0^2-g\phi_0},\ \ \lambda_3=\frac{u_0}{v_0}. \tag{2.8}$$

Then the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ of $\mathcal{E}_2^{-1}\mathcal{E}_1$ are all positive under assumption (2.3). Now we introduce the new variables

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}=P^{-1}\begin{pmatrix}u\\ v\\ \phi\end{pmatrix}=\begin{pmatrix}(v_0u-u_0v+\kappa_0\phi)\frac{1}{2\kappa_0}\\ (v_0u-u_0v-\kappa_0\phi)\frac{-1}{2\kappa_0}\\ (u_0u+v_0v+g\phi)\frac{v_0}{u_0^2+v_0^2}\end{pmatrix}; \tag{2.9}$$

multiplying both sides of (2.5) by P^{-1} gives

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}_y+\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}_x=P^{-1}\mathcal{E}_2^{-1}F, \tag{2.10}$$

which is a fully decoupled system. Therefore, from the general Kreiss-Lopatinskii theory for hyperbolic boundary value problems (see e.g. [Kre70, Lop70] or Chapter 4 in [BS07]), in order to solve the fully decoupled system (2.10) in (ξ, η, ζ), it is necessary and sufficient to specify the boundary conditions for (ξ, η, ζ) at $x = 0$ since λ_1, λ_2 and λ_3 are all positive. That is equivalent to specifying the boundary conditions for $U = (u, v, \phi)$ at $x = 0$.

In conclusion, we will choose to specify the boundary conditions for $U = (u, v, \phi)$ at $x = 0$ and $y = 0$ in the supercritical case corresponding to assumption (2.3).

2.2. **The mixed case.** There are two sub-cases in the mixed case, corresponding to

$$u_0^2 < g\phi_0, \;\; v_0^2 > g\phi_0. \tag{2.11}$$

and

$$u_0^2 > g\phi_0, \;\; v_0^2 < g\phi_0. \tag{2.12}$$

These two cases are similar, we will only consider the first one.

Under assumption (2.11), the calculations (2.4)-(2.10) are still valid, but $\mathcal{E}_2$ only has all its eigenvalues positive. Hence, we still choose the y-direction as the time-like direction, and thus we need to specify the boundary conditions at $y = 0$ (time-like initial conditions), as in the supercritical case. Of course, we also have (2.10), but the difference here is that only λ_1, λ_3 are positive while λ_2 is negative under assumption (2.11). Hence in order to solve system (2.10) in (ξ, η, ζ), it is necessary and sufficient to specify the boundary conditions for ξ and ζ at $x = 0$ since λ_1 and λ_3 are positive and the boundary condition for η at $x = L_1$ since λ_2 is negative (see e.g. [BS07]).

In conclusion, we choose to specify the boundary conditions for $U = (u, v, \phi)$ at $y = 0$, and the boundary conditions for $\xi \simeq v_0 u - u_0 v + \kappa_0 \phi$ and $\zeta \simeq u_0 u + v_0 v + g\phi$ at $x = 0$, and a boundary condition for $\eta \simeq v_0 u - u_0 v - \kappa_0 \phi$ at $x = L_1$ under assumption (2.11).

For the other case, corresponding to (2.12), we choose, with similar arguments, to specify the boundary conditions for $U = (u, v, \phi)$ at $x = 0$, the boundary conditions for $\eta \simeq v_0 u - u_0 v - \kappa_0 \phi$ and $\zeta \simeq u_0 u + v_0 v + g\phi$ at $y = 0$, and a boundary condition for $\xi \simeq v_0 u - u_0 v + \kappa_0 \phi$ at $y = L_2$.

2.3. **The fully hyperbolic subcritical case.** The fully hyperbolic subcritical case corresponds to the case where

$$u_0^2 < g\phi_0, \;\; v_0^2 < g\phi_0. \tag{2.13}$$

Since we are still under the assumption $u_0^2 + v_0^2 - g\phi_0 > 0$, the system (2.5) is still fully hyperbolic. But now $\lambda_1, \lambda_2 < 0$ and $\lambda_3 > 0$. Therefore, by analogy with the mixed case and due to the energy estimates justified in Section 6, we choose to specify the boundary conditions for $\xi \simeq v_0 u - u_0 v + \kappa_0 \phi$ and $\zeta \simeq u_0 u + v_0 v + g\phi$ at $x = 0$, for $\eta \simeq v_0 u - u_0 v - \kappa_0 \phi$ at $x = L_1$, and the boundary conditions for $\eta \simeq v_0 u - u_0 v - \kappa_0 \phi$ and $\zeta \simeq u_0 u + v_0 v + g\phi$ at $y = 0$, and for $\xi \simeq v_0 u - u_0 v + \kappa_0 \phi$ at $y = L_2$.

We now consider the situation when $u_0^2 + v_0^2 - g\phi_0$ is negative, which we call the mixed subcritical case. We set $\kappa_1 = \sqrt{g(g\phi_0 - u_0^2 - v_0^2)/\phi_0}$.

2.4. **The mixed subcritical case.** The mixed subcritical case corresponds to the case where

$$u_0^2 + v_0^2 < g\phi_0, \tag{2.14}$$

which implies that

$$u_0^2 < g\phi_0, \;\; v_0^2 < g\phi_0. \tag{2.15}$$

We notice that (2.7) still holds while P is a complex matrix and λ_1, λ_2 are complex numbers, which indicates that there are some elliptic modes hidden in (2.2). In order to avoid computations with complex numbers and to take advantage of the regularity results valid for elliptic systems (see Sec. 7), we now define P, ξ, η, ζ by setting

$$P^{-1} = \begin{pmatrix} v_0 & -u_0 & 0 \\ 0 & 0 & \kappa_1 \\ u_0 & v_0 & g \end{pmatrix}, \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = P^{-1} \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} v_0 u - u_0 v \\ \kappa_1 \phi \\ u_0 u + v_0 v + g\phi \end{pmatrix};$$

we also denote $S_0 = diag(1, 1, g/\phi_0)$. Then multiplying to the left both sides of (2.2) by $(u_0^2 + v_0^2)P^t S_0$ gives

$$\begin{pmatrix} u_0 & \frac{gv_0}{\kappa_1} & 0 \\ \frac{gv_0}{\kappa_1} & -u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}_x + \begin{pmatrix} v_0 & -\frac{gu_0}{\kappa_1} & 0 \\ -\frac{gu_0}{\kappa_1} & -v_0 & 0 \\ 0 & 0 & v_0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}_y = (u_0^2 + v_0^2)P^t S_0 F. \tag{2.16}$$

We see from (2.16) that the equation satisfied by ζ is decoupled, and is a transport (hyperbolic) equation. Therefore in order to solve for ζ, we need to assign the boundary conditions for ζ at $x = 0$ and $y = 0$ since u_0 and v_0 are both positive.

From (2.16) again, the equations satisfied by ξ and η read

$$\begin{pmatrix} u_0 & \frac{gv_0}{\kappa_1} \\ \frac{gv_0}{\kappa_1} & -u_0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}_x + \begin{pmatrix} v_0 & -\frac{gu_0}{\kappa_1} \\ -\frac{gu_0}{\kappa_1} & -v_0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}_y = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2.17}$$

where f_1, f_2 are the first two components of $(u_0^2 + v_0^2)P^t S_0 F$. The equations (2.17) are elliptic since the four constants $u_0, v_0, gv_0/\kappa_1, -gu_0/\kappa_1$ satisfy condition (7.2) (see Sec. 7). Hence, by Theorem 7.1 and Remark 7.1, in order to solve (2.17) for ξ and η, we can assign different kinds of boundary conditions for ξ and η as long as they satisfy condition (7.1). However, as we will see in Section 8, by taking into account the energy estimates, only some of the boundary conditions satisfying (7.1) are suitable for the well-posedness. In this case we will assign the boundary conditions for ξ at $x = 0$ and $y = 0$, and the boundary conditions for η at $x = L_1$ and $y = L_2$.

In conclusion, we choose to specify the boundary conditions for $\xi = v_0 u - u_0 v$ and $\zeta = u_0 u + v_0 v + g\phi$ at $x = 0$ and $y = 0$, and the boundary conditions for $\eta \simeq v$ at $x = L_1$ and $y = L_2$ when assumption (2.14) holds.

2.5. **A trace theorem.** Since we have to assign boundary conditions on $\partial\Omega$, we first need to make sure that the desired traces at the boundary make sense. Thus we consider the space:

$$\mathcal{X}(\Omega) = \{U \in H = L^2(\Omega)^3, \mathcal{A}U \in H\},$$

endowed with its natural Hilbert norm $(\|U\|_H^2 + \|\mathcal{A}U\|_H^2)^{1/2}$. Then we have the following trace result.

Proposition 2.1. *If $U = (u, v, \phi)^t \in \mathcal{X}(\Omega)$, then the traces of U are defined on all of $\partial\Omega$, i.e. the traces of U are defined at $x = 0, L_1$, and $y = 0, L_2$, and they belong to the respective spaces $H_y^{-1}(0, L_2)$ and $H_x^{-1}(0, L_1)$. Furthermore the trace operators are linear continuous in the corresponding spaces, e.g., $U \in \mathcal{X}(\Omega) \to U|_{x=0}$ is continuous from $\mathcal{X}(\Omega)$ into $H_y^{-1}(0, L_2)^3$.*

Proof. Since $U \in L^2(\Omega)^3 = L_x^2(0, L_1; L_y^2(0, L_2)^3)$, we see that $U_y = \partial U/\partial y$ belongs to $L_x^2(0, L_1; H_y^{-1}(0, L_2)^3)$. Since $\mathcal{A}U \in L^2(\Omega)^3$, that is

$$\begin{cases} u_0 u_x + v_0 u_y + g\phi_x & \in L_x^2(0, L_1; L_y^2(0, L_2)), \\ u_0 v_x + v_0 v_y + g\phi_y & \in L_x^2(0, L_1; L_y^2(0, L_2)), \\ u_0 \phi_x + v_0 \phi_y + \phi_0(u_x + v_y) & \in L_x^2(0, L_1; L_y^2(0, L_2)), \end{cases} \tag{2.18}$$

we infer from $(2.18)_{1,3}$ that

$$\begin{cases} u_0 u_x + g\phi_x & \in L_x^2(0, L_1; H_y^{-1}(0, L_2)), \\ u_0 \phi_x + \phi_0 u_x & \in L_x^2(0, L_1; H_y^{-1}(0, L_2)). \end{cases} \tag{2.19}$$

Noticing that the determinant of the coefficient matrix in (2.19) is nonzero since we exclude the non-generic cases where $u_0^2 = g\phi_0$, we obtain that $u_x, \phi_x \in L_x^2(0, L_1; H_y^{-1}(0, L_2))$. We also find that $v_x \in L_x^2(0, L_1; H_y^{-1}(0, L_2))$ from $(2.18)_2$. Therefore, we have

$$u_x, v_x, \phi_x \in L_x^2(0, L_1; H_y^{-1}(0, L_2)),$$

and in combination with $u, v, \phi \in L_x^2(0, L_1; L_y^2(0, L_2))$, we obtain that

$$u, v, \phi \in \mathcal{C}_x([0, L_1]; H_y^{-1}(0, L_2)), \tag{2.20}$$

which shows that the traces of u, v, ϕ are well-defined at $x = 0$ and L_1, and belong to $H_y^{-1}(0, L_2)$. The continuity of the corresponding mappings is easy. The proof for the traces at $y = 0$ and L_2 is similar. □

3. The density theorems

In this section, we establish general density theorems regarding functions defined on the domain $\Omega = (0, L_1) \times (0, L_2)$. This theorem will be needed for proving later on that $-A$ is the infinitesimal generator of a semigroup of contraction, where A is the 2D shallow water equations operator $\mathcal{A}$ associated with the suitable boundary conditions.

For λ fixed, $\lambda \in \mathbb{R}, \lambda \neq 0$, we set $T\theta = \theta_y + \lambda\theta_x$, and introduce the function space

$$\mathcal{X}_1(\Omega) = \{\theta \in L^2(\Omega), T\theta = \theta_y + \lambda\theta_x \in L^2(\Omega)\}.$$

We need to show that the smooth functions are dense in $\mathcal{X}_1(\Omega)$. Later on we will prove more involved density theorems, showing that if $u \in \mathcal{X}_1(\Omega)$ vanishes on certain parts of $\partial\Omega$, then u can be approximated in $\mathcal{X}_1(\Omega)$ by smooth functions, vanishing on the same parts of the boundary. For the moment, we prove the following.

Proposition 3.1. *$\mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{X}_1(\Omega)$ is dense in $\mathcal{X}_1(\Omega)$.*

Proof. Using a proper covering of Ω by sets $\mathcal{O}_0, \mathcal{O}_1, \cdots, \mathcal{O}_N$, we consider a partition of unity subordinated to this covering, $1 = \sum_{i=0}^N \psi_i$. Here and again in this section we will use a covering of Ω consisting of $\mathcal{O}_0$, a relatively compact subset of Ω, and of sets $\mathcal{O}_i$ of one of the following types: $\mathcal{O}_i$ is a ball centered at one of the corners of Ω, which does not intersect the two other sides of Ω; or $\mathcal{O}_i$ is a ball centered on one of the sides of Ω which does not intersect any of the three other sides of Ω.

If $\theta \in \mathcal{X}_1(\Omega)$, then $\theta\psi_i \in \mathcal{X}_1(\Omega)$, so that we only need to approximate $\theta\psi_i$ by smooth functions. Here the support of ψ_i is contained in the ball $\mathcal{O}_i$, and we start with considering the ball $\mathcal{O}_0$, relatively compact in Ω, then considering the balls $\mathcal{O}_i$ on the boundary $\partial\Omega$.

For any function v defined in Ω, here and again in the following we denote by $\tilde{v}$ the function equal to v in Ω and to 0 outside Ω. We first consider the case where $\psi_i = \psi_0$, and $\mathcal{O}_i = \mathcal{O}_0$, which is relatively compact in Ω. Let ρ be a mollifier such that $\rho \geq 0, \int \rho = 1$, and ρ has compact support. For $v = \theta\psi_0$, the standard mollifier theory (see e.g. Appendix C in [Eva98]) shows that for $\epsilon \to 0$, we have

$$\begin{cases} \rho_\epsilon * \tilde{v} \to \tilde{v}, \\ \rho_\epsilon * \widetilde{Tv} \to \widetilde{Tv}, \end{cases} \tag{3.1}$$

in $L^2(\mathbb{R}^2)$. Now observing that $T\tilde{v} = \widetilde{Tv}$, we find

$$T(\rho_\epsilon * \tilde{v}) = \rho_\epsilon * T\tilde{v} = \rho_\epsilon * \widetilde{Tv} \to \widetilde{Tv}, \tag{3.2}$$

in $L^2(\mathbb{R}^2)$ as $\epsilon \to 0$. Since $\mathcal{O}_0$ is relatively compact in Ω, for ϵ small enough, $\rho_\epsilon * \tilde{v}$ is supported in Ω. If $\tilde{v}_\epsilon = \rho_\epsilon * \tilde{v}$, then $\tilde{v}_\epsilon\big|_\Omega$ converges to v in $\mathcal{X}_1(\Omega)$ by $(3.1)_1$ and (3.2).

We then consider the case where $\psi_i = \psi_1$, and $\mathcal{O}_i = \mathcal{O}_1$ which is a ball centered at the origin $(0,0)$; the other cases are similar or simpler. Let ρ be the same mollifier as before, but now ρ is compactly supported in $\{x < 0, y < 0\}$. Then for $v = \theta\psi_1$, we observe that

$$T\tilde{v} = \widetilde{Tv} + \mu, \tag{3.3}$$

where μ is a measure supported on $\{x = 0\} \cup \{y = 0\}$. Then by mollification with this ρ (see [Hor65]), we observe that, if $\tilde{v}_\epsilon = \rho_\epsilon * \tilde{v}$, then

$$T\tilde{v}_\epsilon = \rho_\epsilon * \widetilde{Tv} + \rho_\epsilon * \mu, \tag{3.4}$$

where $\rho_\epsilon * \mu$ is supported in Ω^c by the choice of ρ. Then, as $\epsilon \to 0$, $\tilde{v}_\epsilon \to \tilde{v}$ in $L^2(\mathbb{R}^2)$, and

$$\begin{cases} \tilde{v}_\epsilon\big|_\Omega \to v, \text{ in } L^2(\Omega); \\ T\tilde{v}_\epsilon\big|_\Omega \to Tv, \text{ in } L^2(\Omega), \end{cases} \tag{3.5}$$

which shows that $\tilde{v}_\epsilon\big|_\Omega$ converges to v in $\mathcal{X}_1(\Omega)$. □

Remark 3.1. *Proposition 3.1 is also valid with T replaced by any first order differential operator with constant coefficients.*

We are now going to prove the density theorems involving the boundary $\partial\Omega$, and we will successively consider the scalar case and the vector case. Here and throughout this article we denote by $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ the boundaries $x = 0, x = L_1, y = 0, y = L_2$ respectively, and

let Γ be any union of the sets $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. In the scalar case, we introduce the function spaces:

$$\mathcal{X}_\Gamma(\Omega) = \{\theta \in L^2(\Omega), T\theta = \theta_y + \lambda\theta_x \in L^2(\Omega), \theta|_\Gamma = 0\}, \tag{3.6}$$

$$\mathcal{V}_\Gamma(\Omega) = \{\theta \in \mathcal{C}^\infty(\overline{\Omega}), \text{ and } \theta \text{ vanishes in a neighborhood of } \Gamma\}.$$

We then state the density theorem:

Theorem 3.1. *Suppose that $\Gamma = \Gamma_1 \cup \Gamma_3$ and $\lambda > 0$, then we have*

$$\mathcal{V}_\Gamma(\Omega) \cap \mathcal{X}_\Gamma(\Omega) \textit{ is dense in } \mathcal{X}_\Gamma(\Omega).$$

To prove Theorem 3.1, we consider the same partition of unity as before; we are led to approximate functions like $\theta\psi_i$ by functions of $\mathcal{V}_\Gamma(\Omega) \cap \mathcal{X}_\Gamma(\Omega)$. The proof for $\theta\psi_0$ is easy as before, we will then consider other functions $\theta\psi_i$ depending on the type of the support $\mathcal{O}_i$.

In order to obtain the local density results regarding certain parts of the boundaries, we introduce the new coordinate system (x', y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+\lambda^2} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

We denote by Γ_i' the image of Γ_i by this transformation for all $i \in \{1, 2, 3, 4\}$; we also denote by Ω' the image of Ω and denote by θ' the transform of θ. Let $(\alpha_{y'}, y')$ and $(\beta_{y'}, y')$ be the end points of the intersection of Ω' with the line y' fixed, $(\alpha_{y'}, y') \in \Gamma_1'$ or Γ_3', $(\beta_{y'}, y') \in \Gamma_2'$ or Γ_4' (see Figure 3.1).

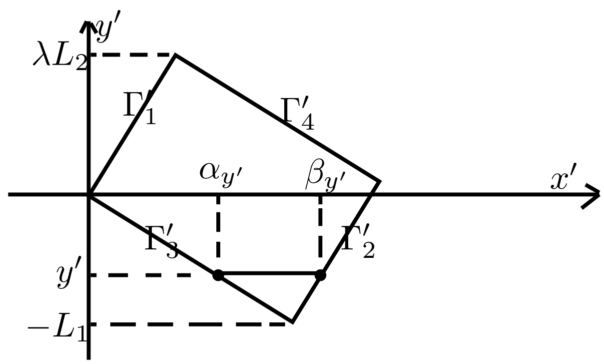

FIGURE 3.1. The domain Ω'

Direct computation shows that

$$\frac{\partial}{\partial x'} = \frac{1}{1+\lambda^2}\left(\frac{\partial}{\partial y} + \lambda\frac{\partial}{\partial x}\right).$$

Therefore, if $\theta \in \mathcal{X}_\Gamma(\Omega)$, we see that θ' and $\theta'_{x'}$ both belong to $L^2(\Omega')$, and $\theta'(\alpha_{y'}, y') = 0$ in the new coordinate system (x', y'). Now for fixed y', we have

$$\theta'(\beta_{y'}, y') = \int_{\alpha_{y'}}^{\beta_{y'}} \theta'_{x'}(\bar{x}, y') d\bar{x}. \tag{3.7}$$

Raising (3.7) to power 2 and integrating with respect to y' from $-L_1$ to λL_2, then using the Cauchy-Schwarz inequality, we finally arrive at

$$\int_{-L_1}^{\lambda L_2} \left|\theta'(\beta_{y'}, y')\right|^2 dy' \le C \left\|\theta'_{x'}\right\|^2_{L^2(\Omega')}, \tag{3.8}$$

where C is a constant independent of θ'. Transforming back into the original system (x, y), (3.8) is equivalent to

$$\int_0^{L_1} \theta(x, L_2)^2 dx + \int_0^{L_2} \theta(L_1, y)^2 dy \le C \left\|T\theta\right\|^2_{L^2(\Omega)}. \tag{3.9}$$

Inequalities (3.8) and (3.9) say that the traces $\theta'\big|_{\Gamma'_2, \Gamma'_4}$ and $\theta\big|_{\Gamma_2, \Gamma_4}$ are L^2 functions with respect to the corresponding segments, e.g. $\theta'\big|_{\Gamma'_4} \in L^2(\Gamma'_4)$.

Now we are ready to show the local density results near the boundaries in the case (3.6). There are eight cases to consider, and we prove them in the following lemmas corresponding to four balls $\mathcal{O}$ centered at the four corners and four balls $\mathcal{O}$ centered on each of the sides.

Lemma 3.1. *Suppose $\theta = \theta(x, y) \in \mathcal{X}_\Gamma(\Omega)$ and $\operatorname{supp} \theta \subset \mathcal{O} \cap \overline{\Omega}$, where $\mathcal{O}$ is a ball centered on Γ_3, which does not intersect $\Gamma_1 \cup \Gamma_2 \cup \Gamma_4$ (see Figure 3.2).*

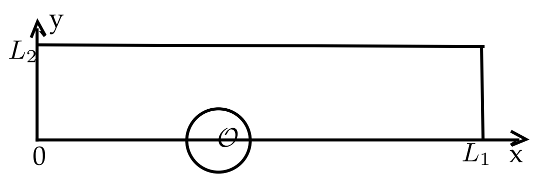

FIGURE 3.2. Ball centered on Γ_3

Then we have

$$T\tilde{\theta} = \widetilde{T\theta}. \tag{3.10}$$

Assume that $\rho \in \mathcal{D}(\mathbb{R}^2)$ has its support compact in $\{|x| < y\}$; then $\rho_\epsilon * \tilde{\theta}$ vanishes in a neighborhood of Γ, and as $\epsilon \to 0$,

$$\begin{cases} (\rho_\epsilon * \tilde{\theta})\big|_\Omega \to \theta, & \text{in } L^2(\Omega), \\ T((\rho_\epsilon * \tilde{\theta})\big|_\Omega) \to T\theta, & \text{in } L^2(\Omega). \end{cases} \tag{3.11}$$

Proof. In the new coordinate system (x', y'), θ' has its support compact in $\mathcal{O}' \cap \overline{\Omega'}$, where $\mathcal{O}'$, the image of $\mathcal{O}$, is a ball centered on Γ'_3, which does not intersect $\Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_4$. We extend θ' by 0 for $x' + \lambda y' < 0$ $(y < 0)$, and denote this extension by $\widetilde{\theta'}$. Noticing that $\theta' = 0$ on $\Gamma'_3 = \{x' + \lambda y' = 0\}$, we then have

$$\widetilde{\theta'}_{x'} = \widetilde{\theta'_{x'}}, \tag{3.12}$$

since $\theta'_{x'}$ belongs to $L^2(\Omega')$. Transforming back to the original system (x, y), (3.12) implies that

$$T\tilde{\theta} = \widetilde{T\theta}. \tag{3.13}$$

Now, for $\epsilon \to 0$, with the same arguments as for (3.1) and (3.2), and using (3.13), we find

$$\begin{cases} \rho_\epsilon * \tilde{\theta} \to \tilde{\theta}, \\ T(\rho_\epsilon * \tilde{\theta}) \to \widetilde{T\theta}, \end{cases} \tag{3.14}$$

in $L^2(\mathbb{R}^2)$.

Since the support of $\rho_\epsilon * \tilde{\theta}$ is contained in $supp(\rho_\epsilon) + supp(\tilde{\theta})$, we infer that $supp(\rho_\epsilon * \tilde{\theta})$ is at a distance away from the x-axis, and included in the interior of Ω. Hence, $\rho_\epsilon * \tilde{\theta}$ vanishes in a neighborhood of Γ by considering the position of $\mathcal{O}$, so that $(\rho_\epsilon * \tilde{\theta})\big|_\Omega \in \mathcal{V}_\Gamma(\Omega)$; furthermore (3.14) implies that

$$\begin{cases} (\rho_\epsilon * \tilde{\theta})\big|_\Omega \to \theta, \\ T((\rho_\epsilon * \tilde{\theta})\big|_\Omega) \to T\theta, \end{cases}$$

in $L^2(\Omega)$ as $\epsilon \to 0$. The proof is complete. □

Proceeding exactly as for Lemma 3.1, we obtain again (3.11) when $\mathcal{O}$ is a ball centered on Γ_1 or at $(0, 0)$; in these cases we respectively take ρ supported in $\{|y| < x\}$ and in $\{0 < \frac{1}{2}x < y < 2x\}$. We thus omit the proof here.

Lemma 3.2. *Suppose $\theta = \theta(x, y) \in \mathcal{X}_\Gamma(\Omega)$ and $supp\,\theta \subset \mathcal{O} \cap \overline{\Omega}$, where $\mathcal{O}$ is a ball centered on Γ_4, which does not intersect $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (see Figure 3.3).*

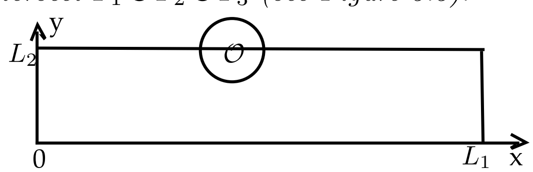

FIGURE 3.3. Ball centered on Γ_4

Then

$$T\tilde{\theta} = \widetilde{T\theta} + \mu, \tag{3.15}$$

where μ is a measure supported by $\Gamma_4 = \{y = L_2\}$. Assume that $\rho \in \mathcal{D}(\mathbb{R}^2)$ has its support compact in $\{|x| < y\}$; then $\rho_\epsilon * \tilde{\theta}$ vanishes in a neighborhood of Γ, and as $\epsilon \to 0$,

$$\begin{cases} (\rho_\epsilon * \tilde{\theta})\big|_\Omega \to \theta, \quad in\ L^2(\Omega), \\ T((\rho_\epsilon * \tilde{\theta})\big|_\Omega) \to T\theta, \quad in\ L^2(\Omega). \end{cases} \tag{3.16}$$

Proof. In the new coordinate system (x', y'), θ' has its support compact in $\mathcal{O}' \cap \overline{\Omega'}$, where $\mathcal{O}'$ is a ball centered on Γ'_4, which does not intersect $\Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$. We extend θ' by 0 for $x' + \lambda y' > (1 + \lambda^2)L_2$ $(y > L_2)$, and denote this extension by $\widetilde{\theta'}$. Observing that $\theta'\big|_{\Gamma'_4} \in L^2(\Gamma'_4)$ (see (3.8) and (3.9)), and that $\theta'_{x'}$ belongs to $L^2(\Omega')$, we thus have

$$\widetilde{\theta'}_{x'} = \widetilde{\theta'_{x'}} + \mu', \tag{3.17}$$

where μ' is a measure supported by $\Gamma_4'\,(=\theta'|_{\Gamma_4'}\delta_{\Gamma_4'})$. Transforming back to the original system (x,y), (3.17) implies that

$$T\tilde{\theta} = \widetilde{T\theta} + \mu, \tag{3.18}$$

where μ is a measure supported by Γ_4. With the choice of ρ, $\rho_\epsilon * \mu$ is supported outside of Ω. Hence, for $\epsilon \to 0$, using the same argument as for (3.4) and (3.5), and noticing (3.18), we obtain

$$\begin{cases} (\rho_\epsilon * \tilde{\theta})\big|_\Omega \to \theta, \text{ in } L^2(\Omega), \\ T((\rho_\epsilon * \tilde{\theta})\big|_\Omega) \to T\theta, \text{ in } L^2(\Omega). \end{cases} \tag{3.19}$$

Finally, $\rho_\epsilon * \tilde{\theta}$ vanishes in a neighborhood of Γ since the support of $\rho_\epsilon * \tilde{\theta}$ is away from Γ. □

Arguing exactly as for Lemma 3.2, we also obtain (3.16) when $\mathcal{O}$ is a ball centered on Γ_2 or at (L_1, L_2), by taking respectively ρ supported in $\{|y| < x\}$ and in $\{0 < \frac{1}{2}x < y < 2x\}$. Thus, we also omit the proof here.

We now turn to the cases where $\mathcal{O}$ is either centered at $(0, L_2)$ or $(L_1, 0)$, and both cases necessitate a specific treatment.

Lemma 3.3. *Suppose $\theta = \theta(x,y) \in \mathcal{X}_\Gamma(\Omega)$ and $supp\,\theta \subset \mathcal{O} \cap \overline{\Omega}$, where $\mathcal{O}$ is a ball centered at $(0, L_2)$, which does not intersect $\Gamma_2 \cup \Gamma_3$ (see Figure 3.4).*

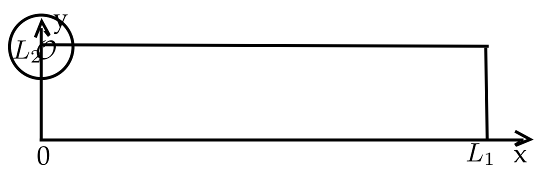

FIGURE 3.4. Ball centered at $(0, L_2)$

Then we have

$$T\tilde{\theta} = \widetilde{T\theta} + \mu, \tag{3.20}$$

where μ is a measure supported by $\Gamma_4 = \{y = L_2\}$. Assume that $\rho \in \mathcal{D}(\mathbb{R}^2)$ has its support compact in $\{0 < \frac{1}{2}x < y < 2x\}$; then $\rho_\epsilon * \tilde{\theta}$ vanishes in a neighborhood of Γ, and as $\epsilon \to 0$,

$$\begin{cases} (\rho_\epsilon * \tilde{\theta})\big|_\Omega \to \theta, \;\; in\ L^2(\Omega), \\ T((\rho_\epsilon * \tilde{\theta})\big|_\Omega) \to T\theta, \;\; in\ L^2(\Omega). \end{cases} \tag{3.21}$$

Proof. In the new coordinate system (x', y'), θ' has its support compact in $\mathcal{O}' \cap \overline{\Omega'}$, where $\mathcal{O}'$ is a ball centered at $(L_2, \lambda L_2)$; it does not intersect $\Gamma_2' \cup \Gamma_3'$. We first extend θ' by 0 for $\lambda x' - y' < 0$ $(x < 0)$, and denote this extension by $\hat{\theta}'$. Noticing that $\theta' = 0$ on $\Gamma_1' = \{\lambda x' - y' = 0\}$, we see that $\widehat{\theta'}_{x'} = \widehat{\theta'_{x'}}$ in the distribution sense in $\widehat{\Omega}' = \{x' + \lambda y' < (1+\lambda^2)L_2\}$ (we also see that $\widehat{\theta'_{x'}} \in L^2(\widehat{\Omega}')$ because $\theta'_{x'} \in L^2(\Omega')$). We then extend $\widehat{\theta}'$ by 0 for

$x'+\lambda y' > (1+\lambda^2)L_2$ ($y > L_2$), and denote this extension by $\widetilde{\theta'}$. Noticing that $\widehat{\theta'}\big|_{\widehat{\Gamma'_4}} \in L^2(\widehat{\Gamma'_4})$ since $\theta'\big|_{\Gamma'_4} \in L^2(\Gamma'_4)$, where $\widehat{\Gamma'_4}$ is the whole line $\{x'+\lambda y' = (1+\lambda^2)L_2\}$, we thus have

$$\widetilde{\theta'}_{x'} = \widetilde{\theta'_{x'}} + \mu', \tag{3.22}$$

where μ' is a measure supported by Γ'_4. The rest of the proof is the same as for Lemma 3.2, we thus omit it here. □

The case where $\mathcal{O}$ is a ball centered at $(L_1, 0)$ is treated similarly as in Lemma 3.3, using a mollifier ρ supported in $\{0 < \frac{1}{2}x < y < 2x\}$. We omit the proof.

We finished considering all the local density results. Thus, at this stage, we are able to prove Theorem 3.1.

Proof of Theorem 3.1. Recall the covering sets $\mathcal{O}_0, \mathcal{O}_1, \cdots, \mathcal{O}_N$ described before and the partition of unity subordinated to this covering $1 = \sum_{i=0}^N \psi_i$ introduced in Proposition 3.1. Given $\theta \in \mathcal{X}_\Gamma(\Omega)$, we set $\theta_i = \psi_i\theta$, so that $\theta_i \in \mathcal{X}_\Gamma(\Omega)$. For any $\epsilon > 0$, By Lemmas 3.1-3.3, we find $\theta_i^\epsilon \in \mathcal{C}^\infty(\overline{\Omega})$ such that

$$\big\|\theta_i^\epsilon - \theta_i\big\|_{L^2(\Omega)} \le \epsilon, \quad \big\|T\theta_i^\epsilon - T\theta_i\big\|_{L^2(\Omega)} \le \epsilon, \tag{3.23}$$

and all θ_i^ϵ's vanish in a neighborhood of Γ. Set $\theta^\epsilon = \sum_{i=0}^N \theta_i^\epsilon$, then θ^ϵ belongs to $\mathcal{C}^\infty(\overline{\Omega})$, and vanishes in a neighborhood of Γ, which shows that θ^ϵ belongs to $\mathcal{V}_\Gamma(\Omega)$.

Noticing that $\theta = \sum_{i=0}^N \psi_i\theta = \sum_{i=0}^N \theta_i$, (3.23) implies

$$\begin{cases} \big\|\theta^\epsilon - \theta\big\|_{L^2(\Omega)} \le \sum_{i=0}^N \big\|\theta_i^\epsilon - \theta_i\big\|_{L^2(\Omega)} \le (N+1)\epsilon, \\ \big\|T\theta^\epsilon - T\theta\big\|_{L^2(\Omega)} \le \sum_{i=0}^N \big\|T\theta_i^\epsilon - T\theta_i\big\|_{L^2(\Omega)} \le (N+1)\epsilon, \end{cases}$$

which shows that θ^ϵ belongs to $\mathcal{X}_\Gamma(\Omega)$ and converges to θ in $\mathcal{X}_\Gamma(\Omega)$ as $\epsilon \to 0$. Hence, the proof is complete. □

Remark 3.2. *Looking back carefully at the proof of Theorem 3.1, we see that Theorem 3.1 is also valid in the following three cases:*

$$\begin{cases} \Gamma = \Gamma_2 \cup \Gamma_4 \text{ and } \lambda > 0, \\ \Gamma = \Gamma_1 \cup \Gamma_4 \text{ and } \lambda < 0, \\ \Gamma = \Gamma_2 \cup \Gamma_3 \text{ and } \lambda < 0, \end{cases} \tag{3.24}$$

provided we choose properly the support of the mollifier. This remark will be useful for Theorem 3.2 and when we study the adjoint A^* of A later on.

We now turn to generalizing Theorem 3.1 to the vector case.

Theorem 3.2. *Let n be a positive integer, and suppose that $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ is a nonsingular diagonal matrix, and M_1, M_2, Q are arbitrary nonsingular matrices satisfying $Q^{-1}\Lambda Q = M_2^{-1}M_1$. Set $\vec{\Gamma} = (\Gamma^1, \cdots, \Gamma^n)$ where*

$$\Gamma^i = \begin{cases} \Gamma_1 \cup \Gamma_3 \text{ or } \Gamma_2 \cup \Gamma_4, & \text{if } \lambda_i > 0, \\ \Gamma_1 \cup \Gamma_4 \text{ or } \Gamma_2 \cup \Gamma_3, & \text{if } \lambda_i < 0; \end{cases}$$

and introduce the function spaces in the vector case:

$$\mathcal{X}_{\vec{\Gamma}}(\Omega) = \{\Theta = (\theta_1, \cdots, \theta_n) \in L^2(\Omega)^n, M_1\Theta_x + M_2\Theta_y \in L^2(\Omega)^n, Q\Theta|_{\vec{\Gamma}} = 0\},$$

$$\mathcal{V}_{\vec{\Gamma}}(\Omega) = \{\Theta \in \mathcal{C}^\infty(\overline{\Omega}), \text{ and } Q\Theta \text{ vanishes in a neighborhood of } \vec{\Gamma}\}.$$

Then we have

$$\mathcal{V}_{\vec{\Gamma}}(\Omega) \cap \mathcal{X}_{\vec{\Gamma}}(\Omega) \text{ is dense in } \mathcal{X}_{\vec{\Gamma}}(\Omega).$$

Proof. Setting $\widehat{\Theta} = Q\Theta$, we first have $\widehat{\Theta}_y + \Lambda\widehat{\Theta}_x = QM_2^{-1}(M_1\Theta_x + M_2\Theta_y)$, and then applying Theorem 3.1 or Remark 3.2 to each component of $\widehat{\Theta}$ and transforming back to Θ, we obtain the result for Theorem 3.2. □

4. The supercritical case

In this section, we consider the stationary 2D shallow water equations operator for the supercritical case; thus we assume that

$$u_0^2 > g\phi_0, \quad v_0^2 > g\phi_0. \tag{4.1}$$

Here, we choose the following boundary conditions according to the discussion in Subsection 2.1:

$$\begin{cases} u = v = \phi = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ u = v = \phi = 0, \text{ on } \Gamma_3 = \{y = 0\}. \end{cases} \tag{4.2}$$

We then define the unbounded operator A on $H = L^2(\Omega)^3$, with $AU = \mathcal{A}U$, $\forall U \in \mathcal{D}(A)$ and

$$\mathcal{D}(A) = \{U \in H = L^2(\Omega)^3, AU \in H, \text{ and } U \text{ satisfies (4.2)}\}.$$

We also introduce the corresponding density boundary conditions:

$$U \text{ vanishes in a neighborhood of } \Gamma_1 \cup \Gamma_3, \tag{4.3}$$

and the function space:

$$\mathcal{V}(\Omega) = \{U \in \mathcal{C}^\infty(\overline{\Omega})^3, \text{ and } U \text{ satisfies (4.3)}\}.$$

Setting $\vec{\Gamma} = (\Gamma_1 \cup \Gamma_3, \Gamma_1 \cup \Gamma_3, \Gamma_1 \cup \Gamma_3)$, we have that $P^{-1}U|_{\vec{\Gamma}} = 0$ is equivalent to (4.2) and $P^{-1}U$ vanishing in a neighborhood of $\vec{\Gamma}$ is the same as (4.3), where P is as in Subsection 2.1. Then we obtain the following result from Theorem 3.2 with $Q = P^{-1}, M_1 = \mathcal{E}_1, M_2 = \mathcal{E}_2, \Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$.

Lemma 4.1. *$\mathcal{V}(\Omega) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$.*

4.1. **Positivity of A and its adjoint A^*.**

We endow the space $H = L^2(\Omega)^3$ with the Hilbert scalar product and norm:

$$\langle U, \overline{U}\rangle_H = \int_\Omega (u\bar{u} + v\bar{v} + \frac{g}{\phi_0}\phi\bar{\phi})dxdy, \quad \|U\|_H = \{\langle U, U\rangle_H\}^{1/2}.$$

Our aim is to prove that A and its adjoint A^* defined below are positive in the sense,

$$\begin{cases} \langle AU, U\rangle_H \geq 0, \quad \forall U \in \mathcal{D}(A), \\ \langle A^*U, U\rangle_H \geq 0, \quad \forall U \in \mathcal{D}(A^*). \end{cases} \tag{4.4}$$

These properties are needed to apply the Hille-Phillips-Yoshida theorem (see [Yos80, HP74]), see below. The result for A is now easy thanks to Lemma 4.1. Indeed the following calculations are easy, for U smooth in $\mathcal{D}(A)$:

$$
\begin{aligned}
\langle AU, U\rangle_H &= \int_\Omega (u_0u_x + v_0u_y + g\phi_x)u + (u_0v_x + v_0v_y + g\phi_y)v \\
&\qquad + \frac{g}{\phi_0}(u_0\phi_x + v_0\phi_y + \phi_0(u_x + v_y))\phi dxdy \\
&= \text{(using integration by parts)} \\
&= \int_0^{L_2} \Big(\frac{u_0}{2}(u^2 + v^2 + \frac{g}{\phi_0}\phi^2) + g\phi u\Big)\Big|_{x=0}^{x=L_1} dy \\
&\qquad + \int_0^{L_1} \Big(\frac{v_0}{2}(u^2 + v^2 + \frac{g}{\phi_0}\phi^2) + g\phi v\Big)\Big|_{y=0}^{y=L_2} dx \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.5}
$$

where I_1 stands for the boundary term at $x = 0$, I_2 at $x = L_1$, I_3 at $y = 0$ and I_4 at $y = L_2$. First, the boundary conditions (4.2) imply that $I_1 = I_3 = 0$. We then rewrite I_2, I_4 as:

$$
\begin{aligned}
I_2 &= \frac{u_0}{2}\big[(u - \frac{g}{u_0}\phi)^2 + g^2(\frac{1}{g\phi_0} - \frac{1}{u_0^2})\phi^2 + v^2\big](L_1, y), \\
I_4 &= \frac{v_0}{2}\big[(v - \frac{g}{v_0}\phi)^2 + g^2(\frac{1}{g\phi_0} - \frac{1}{v_0^2})\phi^2 + u^2\big](x, L_2),
\end{aligned}
$$

which are nonnegative under assumption (4.1). Therefore, we conclude that $\langle AU, U\rangle_H \geq 0$ for U smooth in $\mathcal{D}(A)$, which is also valid for all U in $\mathcal{D}(A)$ thanks to Lemma 4.1.

We now turn to the definition of the formal adjoint A^* of A and its domain $\mathcal{D}(A^*)$, in the sense of the adjoint of a linear unbounded operator (see [Rud91]). For that purpose we first assume that $U \in \mathcal{D}(A)$ and $\overline{U} \in H$ are smooth functions, and then compute

$$
\begin{aligned}
\langle AU, \overline{U}\rangle_H &= \int_\Omega (u_0u_x + v_0u_y + g\phi_x)\bar{u} + (u_0v_x + v_0v_y + g\phi_y)\bar{v} \\
&\qquad + \frac{g}{\phi_0}\big(u_0\phi_x + v_0\phi_y + \phi_0(u_x + v_y)\big)\bar{\phi} dxdy \\
&= J_0 + J_1,
\end{aligned} \tag{4.6}
$$

where J_0 stands for the integral on Ω and J_1 for the integral on $\partial\Omega$. For J_0, we have:

$$
\begin{aligned}
J_0 &= \int_\Omega -(u_0\bar{u}_x + v_0\bar{u}_y + g\bar{\phi}_x)u - (u_0\bar{v}_x + v_0\bar{v}_y + g\bar{\phi}_y)v \\
&\qquad - \frac{g}{\phi_0}\big(u_0\bar{\phi}_x + v_0\bar{\phi}_y + \phi_0(\bar{u}_x + \bar{v}_y)\big)\phi dxdy \\
&= \langle \mathcal{A}^*\overline{U}, U\rangle_H,
\end{aligned}
$$

where $\mathcal{A}^*$ is the differential operator defined as follows:

$$
\mathcal{A}^*\overline{U} = \begin{pmatrix} -u_0\bar{u}_x - v_0\bar{u}_y - g\bar{\phi}_x \\ -u_0\bar{v}_x - v_0\bar{v}_y - g\bar{\phi}_y \\ -u_0\bar{\phi}_x - v_0\bar{\phi}_y - \phi_0(\bar{u}_x + \bar{v}_y) \end{pmatrix}. \tag{4.7}
$$

For J_1, taking into account the boundary conditions (4.2), there remains:

$$J_1 = \int_0^{L_2} \big[u_0(u\bar{u} + v\bar{v} + \phi\bar{\phi}) + g(\phi\bar{u} + u\bar{\phi})]\big](L_1, y)dy \\ + \int_0^{L_1} \big[v_0(u\bar{u} + v\bar{v} + \phi\bar{\phi}) + g(\phi\bar{v} + v\bar{\phi})\big](x, L_2)dx.$$

According to [Rud91], $\mathcal{D}(A^*)$ consists of the $\overline{U}$ in H such that $U \mapsto \langle AU, \overline{U}\rangle_H$ is continuous on $\mathcal{D}(A)$ for the topology (norm) of H. If U is restricted to the class of $\mathcal{C}^\infty$ functions with compact support in Ω, then J_1 vanishes and $U \mapsto J_0$ can only be continuous if $\mathcal{A}^*\overline{U}$ belongs to H. If $\overline{U}$ and $\mathcal{A}^*\overline{U}$ both belong to H, the traces of $\overline{U}$ are well-defined by observing that Proposition 2.1 applies to $\mathcal{A}^*$ as well, and to more general first order linear differential operator with constant coefficients. Hence, the calculations in (4.6) are now valid for any such $\overline{U}$ (and U smooth in $\mathcal{D}(A)$). We now restrict U to the class of $\mathcal{C}^\infty$ functions on $\overline{\Omega}$ which belong to $\mathcal{D}(A)$. Then the expressions above of J_0 and J_1 show that $U \mapsto \langle AU, \overline{U}\rangle_H$ can only be continuous in U for the topology (norm) of H if the following boundary conditions are satisfied:

$$\begin{cases} \bar{u} = \bar{v} = \bar{\phi} = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ \bar{u} = \bar{v} = \bar{\phi} = 0, \text{ on } \Gamma_4 = \{y = L_2\}. \end{cases} \tag{4.8}$$

Conversely, if $\overline{U}, \mathcal{A}^*\overline{U} \in L^2(\Omega)^3$ and conditions (4.8) are satisfied, then the calculations (4.6) are valid, J_1 vanishes, and $U \mapsto \langle AU, \overline{U}\rangle_H$ is continuous on $\mathcal{D}(A)$ for the norm of H. Therefore we conclude that

$$\mathcal{D}(A^*) = \{\overline{U} \in H = L^2(\Omega)^3, A^*\overline{U} \in H, \text{ and } \overline{U} \text{ satisfies } (4.8)\},$$

where we set $A^*\overline{U} = \mathcal{A}^*\overline{U}$, $\forall \overline{U} \in \mathcal{D}(A^*)$.

Let us introduce the density boundary conditions corresponding to (4.8)

$$U \text{ vanishes in a neighborhood of } \Gamma_2 \cup \Gamma_4, \tag{4.9}$$

and define the following space of smooth function with respect to $\mathcal{D}(A^*)$:

$$\mathcal{V}^*(\Omega) = \{\overline{U} \in \mathcal{C}^\infty(\overline{\Omega})^3, \text{ and } \overline{U} \text{ satisfies } (4.9)\}.$$

Applying Theorem 3.2 as we did for Lemma 4.1 with $\vec{\Gamma} = (\Gamma_2 \cup \Gamma_4, \Gamma_2 \cup \Gamma_4, \Gamma_2 \cup \Gamma_4)$, we obtain

Lemma 4.2. *$\mathcal{V}^*(\Omega) \cap \mathcal{D}(A^*)$ is dense in $\mathcal{D}(A^*)$.*

The proof for the positivity of A^* is similar to the proof for A, we thus omit it here.

5. The mixed case

In this section, we consider the stationary 2D shallow water equations operator for the mixed case. As before, we only consider one case, and thus assume that

$$u_0^2 < g\phi_0, \quad v_0^2 > g\phi_0. \tag{5.1}$$

Here, we choose the following boundary conditions according to the discussion in Subsection 2.2:

$$\begin{cases} v_0u - u_0v + \kappa_0\phi = u_0u + v_0v + g\phi = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ v_0u - u_0v - \kappa_0\phi = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ u = v = \phi = 0, \text{ on } \Gamma_3 = \{y = 0\}, \end{cases} \tag{5.2}$$

and introduce the corresponding density boundary conditions:

$$\begin{cases} v_0u - u_0v + \kappa_0\phi, u_0u + v_0v + g\phi \text{ vanish in a neighborhood of } \Gamma_1, \\ v_0u - u_0v - \kappa_0\phi \text{ vanishes in a neighborhood of } \Gamma_2, \\ U \text{ vanishes in a neighborhood of } \Gamma_3. \end{cases} \tag{5.3}$$

We then define the function spaces

$$\mathcal{D}(A) = \{U \in H = L^2(\Omega)^3, \mathcal{A}U \in H, \text{ and } U \text{ satisfies (5.2)}\},$$
$$\mathcal{V}(\Omega) = \{U \in \mathcal{C}^\infty(\overline{\Omega})^3, \text{ and } U \text{ satisfies (5.3)}\},$$

and write $AU = \mathcal{A}U$, $\forall U \in \mathcal{D}(A)$. Setting $\vec{\Gamma} = (\Gamma_1 \cup \Gamma_3, \Gamma_2 \cup \Gamma_3, \Gamma_1 \cup \Gamma_3)$, we have that $P^{-1}U|_{\vec{\Gamma}} = 0$ is equivalent to (5.2) and $P^{-1}U$ vanishing in a neighborhood of $\vec{\Gamma}$ is the same as (5.3), where P is as in Subsection 2.1. Then we obtain the following result from Theorem 3.2 with $Q = P^{-1}, M_1 = \mathcal{E}_1, M_2 = \mathcal{E}_2, \Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$.

Lemma 5.1. *$\mathcal{V}(\Omega) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$.*

5.1. **Positivity of A and its adjoint A^*.** As we already calculated in (4.5), we see that $\langle AU, U\rangle_H = I_1 + I_2 + I_3 + I_4$ holds for $U \in \mathcal{C}^\infty(\overline{\Omega})^3$. If U further satisfies (5.3) (i.e. $U \in \mathcal{V}(\Omega)$), we find $I_3 = 0$, and we rewrite $I_4 = \frac{v_0}{2}\left[(v - \frac{g}{v_0}\phi)^2 + g^2(\frac{1}{g\phi_0} - \frac{1}{v_0^2})\phi^2 + u^2\right](x, L_2)$, which is positive by the assumption $v_0^2 > g\phi_0$. It remains to estimate I_1, I_2. We first introduce the new notations $\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{P}$ such that

$$\hat{P}^{-1} = \begin{pmatrix} v_0 & -u_0 & \kappa_0 \\ v_0 & -u_0 & -\kappa_0 \\ u_0 & v_0 & g \end{pmatrix}, \quad \begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} = \hat{P}^{-1}\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} v_0u - u_0v + \kappa_0\phi \\ v_0u - u_0v - \kappa_0\phi \\ u_0u + v_0v + g\phi \end{pmatrix}. \tag{5.4}$$

Then we compute I_2 as

$$
\begin{aligned}
2I_2 &= [(u,v,\phi)\begin{pmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ g & 0 & \frac{gu_0}{\phi_0}\end{pmatrix}\begin{pmatrix} u \\ v \\ \phi\end{pmatrix}](L_1,y) \\
&= [(\hat{\xi},\hat{\eta},\hat{\zeta})\hat{P}^t\begin{pmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ g & 0 & \frac{gu_0}{\phi_0}\end{pmatrix}\hat{P}\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta}\end{pmatrix}](L_1,y) \\
&= (\text{using that } \hat{\eta} = v_0u - u_0v - \kappa_0\phi \text{ vanishes in a neighborhood of } \{x = L_1\}) \qquad (5.5)\\
&= [(\hat{\xi},0,\hat{\zeta})\hat{P}^t\begin{pmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ g & 0 & \frac{gu_0}{\phi_0}\end{pmatrix}\hat{P}\begin{pmatrix} \hat{\xi} \\ 0 \\ \hat{\zeta}\end{pmatrix}](L_1,y) \\
&= [(\frac{u_0\kappa_0 + gv_0}{2(u_0^2+v_0^2)\kappa_0})\hat{\xi}^2 + (\frac{u_0}{u_0^2+v_0^2})\hat{\zeta}^2](L_1,y) \\
&\geq 0.
\end{aligned}
$$

Similarly, using that $\hat{\xi} = v_0u - u_0v + \kappa_0$ and $\hat{\zeta} = u_0u + v_0v + g\phi$ vanish in a neighborhood of $\{x = 0\}$, we compute $2I_1 = -[((u_0\kappa_0 - gv_0)/(2(u_0^2+v_0^2)\kappa_0))\hat{\eta}^2](0,y) \geq 0$. Therefore, we can conclude that $\langle AU, U\rangle_H \geq 0$ for $U \in \mathcal{V}(\Omega)$, which is also true for all $U \in \mathcal{D}(A)$ thanks to Lemma 5.1.

The formal definition of A^* can be treated similarly as in Subsection 4.1, we thus omit the details. Since we are considering the mixed case, and in order to guarantee that $U \mapsto \langle AU, \overline{U}\rangle_H$ is continuous on $\mathcal{D}(A)$ (see Subsec. 4.1), the following boundary conditions need to be satisfied:

$$
\begin{cases}
v_0\bar{u} - u_0\bar{v} - \kappa_0\bar{\phi} = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\
v_0\bar{u} - u_0\bar{v} + \kappa_0\bar{\phi} = u_0\bar{u} + v_0\bar{v} + g\bar{\phi} = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\
\bar{u} = \bar{v} = \bar{\phi} = 0, \text{ on } \Gamma_4 = \{y = L_2\}.
\end{cases} \tag{5.6}
$$

We conclude that

$$\mathcal{D}(A^*) = \{\overline{U} \in H = L^2(\Omega)^3, A^*\overline{U} \in H, \text{ and } \overline{U} \text{ satisfies (5.6)}\},$$

and write $A^*\overline{U} = \mathcal{A}^*\overline{U}$, $\forall \overline{U} \in \mathcal{D}(A^*)$.

We also introduce the corresponding density boundary conditions:

$$
\begin{cases}
v_0\bar{u} - u_0\bar{v} - \kappa_0\bar{\phi} \text{ vanishes in a neighborhood of } \Gamma_1, \\
v_0\bar{u} - u_0\bar{v} + \kappa_0\bar{\phi}, u_0\bar{u} + v_0\bar{v} + g\bar{\phi} \text{ vanish in a neighborhood of } \Gamma_2, \\
\overline{U} \text{ vanishes in a neighborhood of } \Gamma_4,
\end{cases} \tag{5.7}
$$

and define the corresponding space of smooth function:

$$\mathcal{V}^*(\Omega) = \{\overline{U} \in \mathcal{C}(\overline{\Omega})^3, \text{ and } \overline{U} \text{ satisfies (5.7)}\}.$$

Arguing exactly as in Subsection 4.1, we obtain

Lemma 5.2. *$\mathcal{V}^*(\Omega) \cap \mathcal{D}(A^*)$ is dense in $\mathcal{D}(A^*)$, and A^* is positive in the sense of (4.4).*

6. The fully hyperbolic subcritical case

In this section, we consider the stationary 2D shallow water equations operator for the fully hyperbolic subcritical case. We thus assume that

$$u_0^2 < g\phi_0, \;\; v_0^2 < g\phi_0, \;\; u_0^2 + v_0^2 > g\phi_0. \tag{6.1}$$

Here, we choose the following boundary conditions according to the discussion in Subsection 2.3:

$$\begin{cases} v_0u - u_0v + \kappa_0\phi = u_0u + v_0v + g\phi = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ v_0u - u_0v - \kappa_0\phi = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ v_0u - u_0v - \kappa_0\phi = u_0u + v_0v + g\phi = 0, \text{ on } \Gamma_3 = \{y = 0\}, \\ v_0u - u_0v + \kappa_0\phi = 0, \text{ on } \Gamma_4 = \{y = L_2\}, \end{cases} \tag{6.2}$$

and introduce the corresponding density boundary conditions:

$$\begin{cases} v_0u - u_0v + \kappa_0\phi, u_0u + v_0v + g\phi \text{ vanish in a neighborhood of } \Gamma_1, \\ v_0u - u_0v - \kappa_0\phi \text{ vanishes in a neighborhood of } \Gamma_2, \\ v_0u - u_0v - \kappa_0\phi, u_0u + v_0v + g\phi \text{ vanish in a neighborhood of } \Gamma_3, \\ v_0u - u_0v + \kappa_0\phi \text{ vanishes in a neighborhood of } \Gamma_4. \end{cases} \tag{6.3}$$

We then define the function spaces

$$\begin{aligned} &\mathcal{D}(A) = \{U \in H = L^2(\Omega)^3, \mathcal{A}U \in H, \text{ and } U \text{ satisfies (6.2)}\}, \\ &\mathcal{V}(\Omega) = \{U \in \mathcal{C}^\infty(\overline{\Omega})^3, \text{ and } U \text{ satisfies (6.3)}\}, \end{aligned}$$

and write $AU = \mathcal{A}U, \forall U \in \mathcal{D}(A)$. Setting $\vec{\Gamma} = (\Gamma_1 \cup \Gamma_4, \Gamma_2 \cup \Gamma_3, \Gamma_1 \cup \Gamma_3)$, we have that $P^{-1}U|_{\vec{\Gamma}} = 0$ is equivalent to (6.2) and $P^{-1}U$ vanishing in a neighborhood of $\vec{\Gamma}$ is the same as (6.3), where P is as in Subsection 2.1. Then we obtain the following result from Theorem 3.2 with $Q = P^{-1}, M_1 = \mathcal{E}_1, M_2 = \mathcal{E}_2, \Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$.

Lemma 6.1. *$\mathcal{V}(\Omega) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$.*

6.1. **Positivity of A and its adjoint A^*.** As indicated in Subsection 5.1, we have that $\langle AU, U\rangle_H = I_1 + I_2 + I_3 + I_4$ holds for $U \in \mathcal{C}^\infty(\overline{\Omega})^3$. Suppose that U belongs to $\mathcal{V}(\Omega)$. In order to estimate I_1, I_2, I_3, I_4, we still use the notations $\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{P}$ introduced in Subsection 5.1 (see (5.4)), and I_1, I_2 are both nonnegative since the estimates for I_1, I_2 are exactly the

same as in Subsection 5.1. Proceeding exactly as for I_2, we compute I_4 as:

$$
\begin{aligned}
2I_4 &= [(u,v,\phi)\begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & g & \frac{gv_0}{\phi_0}\end{pmatrix}\begin{pmatrix} u \\ v \\ \phi\end{pmatrix}](x,L_2) \\
&= [(\hat{\xi},\hat{\eta},\hat{\zeta})\hat{P}^t\begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & g & \frac{gv_0}{\phi_0}\end{pmatrix}\hat{P}\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta}\end{pmatrix}](x,L_2) \\
&= (\text{using that } \hat{\xi} = v_0u - u_0v + \kappa_0\phi \text{ vanishes in a neighborhood of } \{y = L_2\}) \\
&= [(0,\hat{\eta},\hat{\zeta})\hat{P}^t\begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & g & \frac{gv_0}{\phi_0}\end{pmatrix}\hat{P}\begin{pmatrix} 0 \\ \hat{\eta} \\ \hat{\zeta}\end{pmatrix}](x,L_2) \\
&= [(\frac{v_0\kappa_0 + gu_0}{2(u_0^2+v_0^2)\kappa_0})\hat{\eta}^2 + (\frac{v_0}{u_0^2+v_0^2})\hat{\zeta}^2](x,L_2) \\
&\geq 0.
\end{aligned} \tag{6.4}
$$

Similarly, using that $\hat{\eta} = v_0u - u_0v - \kappa_0$ and $\hat{\zeta} = u_0u + v_0v + g\phi$ vanish in a neighborhood of $\{y = 0\}$, we compute $2I_3 = -[((v_0\kappa_0 - gu_0)/(2(u_0^2+v_0^2)\kappa_0))\hat{\eta}^2](x,0) \geq 0$. Therefore, we conclude that $\langle AU, U\rangle_H \geq 0$ for $U \in \mathcal{V}(\Omega)$; this implies that $\langle AU, U\rangle_H \geq 0$ also holds for all $U \in \mathcal{D}(A)$ by virtue of Lemma 6.1. Hence A is positive.

Taking similar arguments in Subsection 4.1 and Subsection 5.1, we will obtain the same results for A^* the adjoint operator of A. We thus only state them below without the proof.

In the fully hyperbolic subcritical case, we first introduce the following boundary conditions ,

$$
\begin{cases}
v_0\bar{u} - u_0\bar{v} - \kappa_0\bar{\phi} = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\
v_0\bar{u} - u_0\bar{v} + \kappa_0\bar{\phi} = u_0\bar{u} + v_0\bar{v} + g\bar{\phi} = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\
v_0\bar{u} - u_0\bar{v} + \kappa_0\bar{\phi} = 0, \text{ on } \Gamma_3 = \{y = 0\}, \\
v_0\bar{u} - u_0\bar{v} - \kappa_0\bar{\phi} = u_0\bar{u} + v_0\bar{v} + g\bar{\phi} = 0, \text{ on } \Gamma_4 = \{y = L_2\};
\end{cases} \tag{6.5}
$$

and the corresponding density boundary conditions:

$$
\begin{cases}
v_0\bar{u} - u_0\bar{v} - \kappa_0\bar{\phi} \text{ vanishes in a neighborhood of } \Gamma_1, \\
v_0\bar{u} - u_0\bar{v} + \kappa_0\bar{\phi}, u_0\bar{u} + v_0\bar{v} + g\bar{\phi} \text{ vanish in a neighborhood of } \Gamma_2, \\
v_0\bar{u} - u_0\bar{v} + \kappa_0\bar{\phi} \text{ vanishes in a neighborhood of } \Gamma_3, \\
v_0\bar{u} - u_0\bar{v} - \kappa_0\bar{\phi}, u_0\bar{u} + v_0\bar{v} + g\bar{\phi} \text{ vanish in a neighborhood of } \Gamma_4.
\end{cases} \tag{6.6}
$$

We then define the function spaces:

$$
\begin{aligned}
&\mathcal{D}(A^*) = \{\overline{U} \in H = L^2(\Omega)^3, A^*\overline{U} \in H, \text{ and } \overline{U} \text{ satisfies (6.5)}\}, \\
&\mathcal{V}^*(\Omega) = \{\overline{U} \in \mathcal{C}(\overline{\Omega})^3, \text{ and } \overline{U} \text{ satisfies (6.6)}\},
\end{aligned}
$$

and write $A^*\overline{U} = \mathcal{A}^*\overline{U}$, $\forall \overline{U} \in \mathcal{D}(A^*)$. Then we have the following results.

Lemma 6.2. *$\mathcal{V}^*(\Omega) \cap \mathcal{D}(A^*)$ is dense in $\mathcal{D}(A^*)$, and A^* is positive in the sense of (4.4).*

7. A preliminary regularity theorem

We aim in Section 8 to study the mixed subcritical case which corresponds to

$$u_0^2 + v_0^2 < g\phi_0, \text{ implying } u_0^2 < g\phi_0,\ v_0^2 < g\phi_0,$$

and which embodies some elliptic operators. In this section we introduce some necessary preliminary results, namely we prove a general regularity theorem regarding functions defined on the domain $\Omega = (0, L_1) \times (0, L_2)$. This theorem will be very useful in verifying the validity of the integration by parts in Section 8. Following the notations in Section 3, we introduce the following boundary conditions:

$$\begin{cases} \theta_1 = 0 \text{ on } \Gamma, \\ \theta_2 = 0 \text{ on } \Gamma^c, \end{cases} \tag{7.1}$$

where Γ^c is the complement of Γ with respect to the boundary $\partial\Omega$. Here we choose $\Gamma = \Gamma_1 \cup \Gamma_3$, which is the case considered in Section 8. We define

$$V = \{\Theta = (\theta_1, \theta_2)^t \in H^1(\Omega)^2 \mid \Theta \text{ satisfies (7.1) }\}.$$

Note that $\|\nabla\Theta\|_{L^2(\Omega)^2}$ is a natural norm on V thanks to the Poincaré inequality.

We assume that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are four real constants satisfying

$$\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0, \tag{7.2}$$

and we define the operator $\mathcal{T}$ such that

$$\mathcal{T}\Theta = T_1\Theta_x + T_2\Theta_y, \tag{7.3}$$

where

$$T_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & -\alpha_1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \beta_2 & -\alpha_2 \end{pmatrix}.$$

Notice that $\mathcal{T}$ is symmetric and elliptic, and actually we have the following statement.

Proposition 7.1. $\|\mathcal{T}\Theta\|_{L^2(\Omega)^2}$ *is a norm on V equivalent to the H^1-norm.*

Proof. Let us recall a basic fact from linear algebra. We endow $\mathbb{R}^2$ with its usual dot product and induced norm:

$$x \cdot y = x_1y_1 + x_2y_2, \quad |x|_2 = \sqrt{x_1^2 + x_2^2},$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $\mathbb{T}$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ with T_0 as its matrix representation, where

$$T_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}.$$

Condition (7.2) shows that T_0 is non-singular, which implies that $\mathbb{T}$ is an isomorphism. Hence we have that

$$\frac{1}{c_1}|x|_2 \leq |\mathbb{T}x|_2 \leq c_2|x|_2, \tag{7.4}$$

where c_2 (resp. c_1) is the spectral norm of T_0 (resp. T_0^{-1}), i.e. the square root of the largest eigenvalue of the positive-definite matrix $T_0^tT_0$ (resp. $T_0^{-t}T_0^{-1}$).

We now compute

$$\begin{aligned}
\|\mathcal{T}\Theta\|^2_{L^2(\Omega)^2} &= \int_\Omega (\Theta_x^t T_1^t + \Theta_y^t T_2^t)(T_1\Theta_x + T_2\Theta_y)dxdy \\
&= \int_\Omega (\alpha_1^2+\beta_1^2)(\theta_{1x}^2+\theta_{2x}^2) + (\alpha_2^2+\beta_2^2)(\theta_{1y}^2+\theta_{2y}^2) \\
&\quad + 2(\alpha_1\alpha_2+\beta_1\beta_2)(\theta_{1x}\theta_{1y}+\theta_{2x}\theta_{2y}) \\
&\quad + 2(\alpha_2\beta_1-\alpha_1\beta_2)(\theta_{2x}\theta_{1y}-\theta_{1x}\theta_{2y})dxdy.
\end{aligned} \tag{7.5}$$

To dispense with the last term in the integral of (7.5), we use a result from [Gri85] (see Lemma 4.3.1.3), which implies:

Lemma 7.1. *The identity*

$$\int_\Omega \theta_{2x}\theta_{1y}dxdy = \int_\Omega \theta_{1x}\theta_{2y}dxdy$$

holds for all $\Theta = (\theta_1,\theta_2)^t \in H^1(\Omega)^2$ satisfying (7.1) (i.e. $\Theta \in V$).

Thanks to Lemma 7.1, (7.5) gives

$$\begin{aligned}
\|\mathcal{T}\Theta\|^2_{L^2(\Omega)^2} &= \int_\Omega (\alpha_1\theta_{1x}+\alpha_2\theta_{1y})^2 + (\beta_1\theta_{1x}+\beta_2\theta_{1y})^2 \\
&\quad + (\alpha_1\theta_{2x}+\alpha_2\theta_{2y})^2 + (\beta_1\theta_{2x}+\beta_2\theta_{2y})^2dxdy \\
&= \int_\Omega |\mathbb{T}\nabla\theta_1|_2^2 + |\mathbb{T}\nabla\theta_2|_2^2dxdy.
\end{aligned} \tag{7.6}$$

With the help of (7.4), (7.6) immediately implies that

$$\frac{1}{c_1}\|\nabla\Theta\|_{L^2(\Omega)^2} \le \|\mathcal{T}\Theta\|_{L^2(\Omega)^2} \le c_2\|\nabla\Theta\|_{L^2(\Omega)^2}.$$

Hence Proposition 7.1 follows. □

The regularity result is the following existence theorem.

Theorem 7.1. *For every given $F = (f_1,f_2)^t \in L^2(\Omega)^2$, the problem $\mathcal{T}\Theta = F$ has a unique solution $\Theta \in V$.*

Proof. Let $l(\overline{\Theta}) = \langle F, \mathcal{T}\overline{\Theta}\rangle$ and $a(\Theta,\overline{\Theta}) = \langle \mathcal{T}\Theta, \mathcal{T}\overline{\Theta}\rangle_{L^2(\Omega)^2}$; then it is not hard to check that the linear form l is continuous on V, and the bilinear form a is continuous on $V\times V$. Observing that the form a is coercive on V thanks to Proposition 7.1, we obtain that there exists a unique $\Theta \in V$ such that

$$a(\Theta,\overline{\Theta}) = l(\overline{\Theta}), \tag{7.7}$$

for all $\overline{\Theta} \in V$ thanks to the Lax-Milgram Theorem.

In order to interpret (7.7) in the distribution sense, we need the following lemma.

Lemma 7.2. *For every given $\Psi = (\psi_1,\psi_2)^t \in \mathcal{D}(\Omega)^2$, the problem $\mathcal{T}\overline{\Theta} = \Psi$ has a unique solution $\overline{\Theta} \in V$.*

The proof of Lemma 7.2 will be given below, and we continue with the proof of Theorem 7.1. For any $\Psi \in \mathcal{D}(\Omega)^2$, we find $\overline{\Theta} \in V$ such that $\mathcal{T}\overline{\Theta} = \Psi$ by virtue of Lemma 7.2, and then (7.7) shows that $\mathcal{T}\Theta = F$ in the distribution sense in Ω, and we conclude that $\mathcal{T}\Theta = F$ in $L^2(\Omega)^2$ since $\mathcal{D}(\Omega)^2$ is dense in $L^2(\Omega)^2$. □

Proof of Lemma 7.2. Without loss of generality, we first assume that,

$$\alpha_2\beta_1 - \alpha_1\beta_2 = 1. \tag{7.2$'$}$$

Let $\Psi = (\psi_1, \psi_2)^t \in \mathcal{D}(\Omega)^2$: we look for $\overline{\Theta} \in V$ satisfying $\mathcal{T}\overline{\Theta} = \Psi$. In order to find such $\overline{\Theta} \in V$, we introduce a new coordinate system (x', y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \beta_2 & -\beta_1 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta_2 x - \beta_1 y \\ \alpha_2 x - \alpha_1 y \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\alpha_1 & \beta_1 \\ -\alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\alpha_1 x' + \beta_1 y' \\ -\alpha_2 x' + \beta_2 y' \end{pmatrix}.$$

We denote by Γ'_i the image of Γ_i by this transformation for all $i \in \{1, 2, 3, 4\}$, and denote by $\Omega', \Gamma', \theta', \Psi'$ and the gradient ∇' the transforms of $\Omega, \Gamma, \theta, \Psi$ and the gradient ∇ respectively. Now, direct computation gives

$$\nabla\theta = \begin{pmatrix} \beta_2 & \alpha_2 \\ -\beta_1 & -\alpha_1 \end{pmatrix} \nabla'\theta'. \tag{7.8}$$

In the new coordinate system (x', y'), the boundary conditions (7.1) read

$$\begin{cases} \theta'_1 = 0 \text{ on } \Gamma', \\ \theta'_2 = 0 \text{ on } \Gamma'^c, \end{cases} \tag{7.1$'$}$$

where $\Gamma' = \Gamma'_1 \cup \Gamma'_3$. We also denote by V' the function space corresponding to V:

$$V' = \{\Theta' = (\theta'_1, \theta'_2)^t \in H^1(\Omega')^2 \mid \Theta' = (\theta'_1, \theta'_2)^t \text{ satisfies } (7.1') \ \}.$$

In the new coordinate system (x', y'), the operator $\mathcal{T}$ reads

$$\begin{aligned}
\mathcal{T}\Theta' &= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \nabla\theta_1 + \begin{pmatrix} \beta_1 & \beta_2 \\ -\alpha_1 & -\alpha_2 \end{pmatrix} \nabla\theta_2 \\
&= (\text{using } (7.8)) \\
&= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \beta_2 & \alpha_2 \\ -\beta_1 & -\alpha_1 \end{pmatrix} \nabla'\theta'_1 + \begin{pmatrix} \beta_1 & \beta_2 \\ -\alpha_1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} \beta_2 & \alpha_2 \\ -\beta_1 & -\alpha_1 \end{pmatrix} \nabla'\theta'_2 \\
&= (\text{using assumption } (7.2')) \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \nabla'\theta'_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \nabla'\theta'_2 \\
&= \begin{pmatrix} -\theta'_{1x'} + \theta'_{2y'} \\ \theta'_{1y'} + \theta'_{2x'} \end{pmatrix},
\end{aligned} \tag{7.9}$$

and of course, we have $\Psi' \in \mathcal{D}(\Omega')$. We are seeking $\overline{\Theta}' \in V'$ satisfying $\mathcal{T}\overline{\Theta}' = \Psi'$, that is

$$\begin{cases} -\bar{\theta}'_{1x'} + \bar{\theta}'_{2y'} = \psi'_1, \\ \bar{\theta}'_{1y'} + \bar{\theta}'_{2x'} = \psi'_2. \end{cases} \tag{7.10}$$

Differentiating $(7.10)_1$ with respect to x' and $(7.10)_2$ with respect to y', and subtracting these two equations, we find the elliptic equation

$$\Delta'\bar{\theta}'_1 = -\psi'_{1x'} + \psi'_{2y'}, \tag{7.11}$$

where Δ' denotes the Laplace operator in the new coordinate system (x', y'). We associate to equation (7.11) the boundary conditions

$$\bar{\theta}'_1 = 0 \text{ on } \Gamma', \tag{7.12}$$

which is already contained in (7.1′). For the boundary Γ'^c, suitable boundary conditions can be obtained as follows. We first denote by ν'_i the unit normal vector to Γ'_i, and τ'_i the unit tangent vector on Γ'_i, for all $i \in \{1, 2, 3, 4\}$. On $\Gamma'_2 = \{-\alpha_1 x' + \beta_1 y' = L_1\}$, we have $\partial\bar{\theta}'_2/\partial\tau'_2 = 0$ since $\bar{\theta}'_2 = 0$ on Γ'_2. Noticing that (β_1, α_1) is parallel to τ'_2, we find

$$0 = (\beta_1, \alpha_1)\begin{pmatrix} \bar{\theta}'_{2x'} \\ \bar{\theta}'_{2y'} \end{pmatrix} = \beta_1\bar{\theta}'_{2x'} + \alpha_1\theta'_{2y'}, \text{ on } \Gamma'_2. \tag{7.13}$$

Multiplying $(7.10)_1$ by α_1, (7.10) by β_1, and adding these two equations, we find

$$-\alpha_1\bar{\theta}'_{1x'} + \beta_1\bar{\theta}'_{1y'} + \alpha_1\bar{\theta}'_{2y'} + \beta_1\bar{\theta}'_{2x'} = \alpha_1\psi'_1 + \beta_1\psi'_2,$$

which, by using (7.13) and noticing that $\Psi' = (\psi'_1, \psi'_2)$ vanishes on Γ'_2, implies that

$$(-\alpha_1, \beta_1)\begin{pmatrix} \bar{\theta}'_{1x'} \\ \bar{\theta}'_{1y'} \end{pmatrix} = -\alpha_1\bar{\theta}'_{1x'} + \beta_1\bar{\theta}'_{1y'} = 0, \text{ on } \Gamma'_2. \tag{7.14}$$

Observing that ν'_2 is parallel to $(-\alpha_1, \beta_1)$, we then associate to (7.11) the following boundary condition on Γ'_2:

$$\frac{\partial\bar{\theta}'_1}{\partial\nu'_2} = 0, \text{ on } \Gamma'_2, \tag{7.15}$$

which is equivalent to (7.14). On $\Gamma'_4 = \{-\alpha_2 x' + \beta_2 y' = L_2\}$, we have $\partial\bar{\theta}'_2/\partial\tau'_4 = 0$ since $\bar{\theta}'_2 = 0$ on Γ'_4. Noticing that (β_2, α_2) is parallel to τ'_4, and ν'_4 is parallel to $(-\alpha_2, \beta_2)$, similar computations show that we need to associate to (7.11) the following boundary condition on Γ'_4:

$$\frac{\partial\bar{\theta}'_1}{\partial\nu'_4} = 0, \text{ on } \Gamma'_4. \tag{7.16}$$

The existence and uniqueness of a solution $\bar{\theta}'_1 \in H^1(\Omega')$ of (7.11)-(7.12) and (7.15)-(7.16) follows from Lemma 4.4.3.1 of [Gri85] with $\Omega = \Omega', \mathscr{D} = \Gamma', \mathscr{N} = \Gamma'^c, \beta_j = 0, f = -\psi_{1x'} + \psi_{2y'}$.

Following the arguments for $\bar{\theta}_1'$, we find the following equation and boundary conditions for $\bar{\theta}_2$:

$$\begin{cases} \Delta' \bar{\theta}_2' = \psi'_{1y'} + \psi'_{2x'}, \\ \bar{\theta}_2' = 0, \text{ on } \Gamma'^c = \Gamma_2' \cup \Gamma_4', \\ \frac{\partial \bar{\theta}_2'}{\partial \nu_1'} = 0, \text{ on } \Gamma_1', \\ \frac{\partial \bar{\theta}_2'}{\partial \nu_3'} = 0, \text{ on } \Gamma_3'. \end{cases} \tag{7.17}$$

We thus also have a unique solution $\bar{\theta}_2' \in H^1(\Omega')$ of (7.17) thanks to Lemma 4.4.3.1 of [Gri85] again.

In conclusion, in the new coordinate system (x', y'), we find a unique solution $\overline{\Theta}'$ that belongs to V', and solves the problem $\mathcal{T}\overline{\Theta}' = \Psi'$. Transforming back to the original coordinate system (x, y), we obtain $\overline{\Theta} \in V$ satisfying $\mathcal{T}\overline{\Theta} = \Psi$. Hence, the proof of Lemma 7.2 is complete.

□

Remark 7.1. *Theorem 7.1 is still true, if we choose any other boundary* Γ *except* $\emptyset$ *and* $\cup_{i=1}^4 \Gamma_i$. *The proof is exactly the same.*

We will now develop a similar result for $\mathcal{T}^*$ the adjoint operator of $\mathcal{T}$. The adjoint $\mathcal{T}^*$ is formally calculated as usual. For smooth functions Θ and $\overline{\Theta}$, we compute

$$\begin{aligned} \int_\Omega \mathcal{T}\Theta \cdot \overline{\Theta} dxdy &= \int_\Omega \overline{\Theta}^t T_1 \Theta_x + \overline{\Theta}^t T_2 \Theta_y dxdy \\ &= \text{(using integration by parts)} \\ &= -\int_\Omega \Theta^t T_1 \overline{\Theta}_x + \Theta^t T_2 \overline{\Theta}_y dxdy \\ &\quad + \int_0^{L_2} \overline{\Theta}^t T_1 \Theta \Big|_{x=0}^{x=L_1} dy + \int_0^{L_1} \overline{\Theta}^t T_1 \Theta \Big|_{y=0}^{y=L_2} dx \\ &= \text{(using the boundary conditions (7.1))} \\ &= \int_\Omega \mathcal{T}^* \overline{\Theta} \cdot \Theta dxdy \\ &\quad + \int_0^{L_2} (\alpha_1 \bar{\theta}_1 + \beta_1 \bar{\theta}_2)\theta_1 \Big|_{x=L_1} - (\beta_1 \bar{\theta}_1 - \alpha_1 \bar{\theta}_2)\theta_2 \Big|_{x=0} dy \\ &\quad + \int_0^{L_1} (\alpha_2 \bar{\theta}_1 + \beta_2 \bar{\theta}_2)\theta_1 \Big|_{y=L_2} - (\beta_2 \bar{\theta}_1 - \alpha_2 \bar{\theta}_2)\theta_2 \Big|_{y=0} dx, \end{aligned} \tag{7.18}$$

where $\mathcal{T}^*\overline{\Theta} = -T_1\overline{\Theta}_x - T_2\overline{\Theta}_y = -\mathcal{T}\overline{\Theta}$. Hence, we introduce the following boundary conditions corresponding to the adjoint operator $\mathcal{T}^*$ (dropping the bars):

$$\begin{cases} \beta_1\theta_1 - \alpha_1\theta_2 = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ \alpha_1\theta_1 + \beta_1\theta_2 = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ \beta_2\theta_1 - \alpha_2\theta_2 = 0, \text{ on } \Gamma_3 = \{y = 0\}, \\ \alpha_2\theta_1 + \beta_2\theta_2 = 0, \text{ on } \Gamma_4 = \{y = L_2\}; \end{cases} \tag{7.19}$$

and the function space

$$\overline{V} = \{\Theta = (\theta_1, \theta_2)^t \in H^1(\Omega)^2 \mid \Theta \text{ satisfies (7.19) }\}.$$

The results for $\mathcal{T}^*$ associated with $\overline{V}$ are the same as for $\mathcal{T}$ associated with V. We state them as follows.

Proposition 7.2. $\left\|\mathcal{T}^*\Theta\right\|_{L^2(\Omega)^2}$ *is a norm on* $\overline{V}$ *equivalent to the* H^1*-norm.*

The proof of Proposition 7.2 is the same as the proof of Proposition 7.1 since Lemma 7.1 also holds for all $\Theta = (\theta_1, \theta_2)^t \in H^1(\Omega)^2$ satisfying (7.19) (i.e. $\Theta \in \overline{V}$).

Theorem 7.2. *For every given* $F = (f_1, f_2)^t \in L^2(\Omega)^2$*, the problem* $\mathcal{T}^*\Theta = F$ *has a unique solution* $\Theta \in \overline{V}$*.*

The proof of Theorem 7.2 is also the same as the proof of Theorem 7.1 using the following lemma which replaces Lemma 7.2.

Lemma 7.3. *For every given* $\Psi = (\psi_1, \psi_2)^t \in \mathcal{D}(\Omega)^2$*, the problem* $\mathcal{T}^*\overline{\Theta} = \Psi$ *possesses a (possibly non-unique) solution* $\overline{\Theta} \in \overline{V}$*.*

Proof. To show the result, we introduce a suitable change of variables (different from the previous one) which makes the boundary conditions (7.19) simpler. We set

$$\Xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\bar{\theta}_1 + \beta_1\bar{\theta}_2 \\ \beta_1\bar{\theta}_1 - \alpha_1\bar{\theta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & -\alpha_1 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{pmatrix} = T_1\overline{\Theta},$$

and find by direct computation

$$\beta_2\bar{\theta}_1 - \alpha_2\bar{\theta}_2 = \mu_1\xi_1 + \mu_2\xi_2, \ \ \alpha_2\bar{\theta}_1 + \beta_2\bar{\theta}_2 = \mu_2\xi_1 - \mu_1\xi_2,$$

where the constants μ_1 and μ_2 are given by

$$\mu_1 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1^2 + \beta_1^2}, \ \ \mu_2 = \frac{\alpha_1\beta_1 + \alpha_2\beta_2}{\alpha_1^2 + \beta_1^2}.$$

Thus the boundary conditions (7.19) are equivalent to

$$\begin{cases} \xi_2 = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ \xi_1 = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ \mu_1\xi_1 + \mu_2\xi_2 = 0, \text{ on } \Gamma_3 = \{y = 0\}, \\ \mu_2\xi_1 - \mu_1\xi_2 = 0, \text{ on } \Gamma_4 = \{y = L_2\}. \end{cases} \tag{7.19$^\flat$}$$

With the new variables, we define the operator $\mathcal{P}$ such that $\mathcal{P}\Xi = \mathcal{T}^*\overline{\Theta}$, that is

$$\begin{aligned} \mathcal{P}\Xi = \mathcal{T}^*\overline{\Theta} &= -T_1\overline{\Theta}_x - T_2\overline{\Theta}_y = -\Xi - T_2T_1^{-1}\Xi \\ &= -\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x - \begin{pmatrix} \mu_2 & -\mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_y \\ &= -\begin{pmatrix} 1 & \mu_2 \\ 0 & \mu_1 \end{pmatrix} \nabla\xi_1 - \begin{pmatrix} 0 & -\mu_1 \\ 1 & \mu_2 \end{pmatrix} \nabla\xi_2. \end{aligned} \tag{7.20}$$

The goal now becomes the following: given $\Psi = (\psi_1, \psi_2)^t \in \mathcal{D}(\Omega)^2$, we look for $\Xi \in \overline{V}^\flat$ such that $\mathcal{P}\Xi = \Psi$, where the function space $\overline{V}^\flat$ is defined as

$$\overline{V}^\flat = \{\Xi = (\xi_1, \xi_2)^t \in H^1(\Omega)^2 \mid \Xi \text{ satisfies } (7.19^\flat)\ \}.$$

Noticing that μ_1 is nonzero because of assumption (7.2), we can assume without loss of generality that $\mu_1 = 1$. Following the proof of Lemma 7.2, we then introduce a new coordinate system (x', y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mu_2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mu_2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

We denote by ∇' the gradient in the new coordinate system (x', y') In the new coordinate system (x', y'), the gradient ∇' and the operator $\mathcal{P}$ read

$$\nabla \xi = \begin{pmatrix} 1 & -\mu_2 \\ 0 & 1 \end{pmatrix} \nabla' \xi; \tag{7.21}$$

$$\begin{aligned} \mathcal{P}\Xi &= -\begin{pmatrix} 1 & \mu_2 \\ 0 & 1 \end{pmatrix} \nabla \xi_1 - \begin{pmatrix} 0 & -1 \\ 1 & \mu_2 \end{pmatrix} \nabla \xi_2 \\ &= -\begin{pmatrix} 1 & \mu_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\mu_2 \\ 0 & 1 \end{pmatrix} \nabla' \xi_1 - \begin{pmatrix} 0 & -1 \\ 1 & \mu_2 \end{pmatrix} \begin{pmatrix} 1 & -\mu_2 \\ 0 & 1 \end{pmatrix} \nabla' \xi_2 \\ &= \begin{pmatrix} -\xi_{1x'} + \xi_{2y'} \\ -\xi_{1y'} - \xi_{2x'} \end{pmatrix}; \end{aligned} \tag{7.22}$$

and the boundary conditions $(7.19^\flat)$ read

$$\begin{cases} \xi_2 = 0, \text{ on } \Gamma_1', \\ \xi_1 = 0, \text{ on } \Gamma_2', \\ \xi_1 + \mu_2 \xi_2 = 0, \text{ on } \Gamma_3', \\ \mu_2 \xi_1 - \xi_2 = 0, \text{ on } \Gamma_4'. \end{cases} \tag{7.19$^\sharp$}$$

Finally we define the function space

$$\overline{V}^\sharp = \{\Xi = (\xi_1, \xi_2)^t \in H^1(\Omega')^2 \mid \Xi \text{ satisfies } (7.19^\sharp)\ \}.$$

We are now seeking $\Xi \in \overline{V}^\sharp$ satisfying $\mathcal{P}\Xi = \Psi$, that is

$$\begin{cases} -\xi_{1x'} + \xi_{2y'} = \psi_1, \\ -\xi_{1y'} - \xi_{2x'} = \psi_2. \end{cases} \tag{7.23}$$

We now have two cases to consider. First, if $\mu_2 = 0$, then the boundary conditions $(7.19^\sharp)$ are simpler, and similar to the boundary conditions $(7.1')$. Hence, we obtain a unique solution $\Xi \in \overline{V}^\flat$ that solves the problem $\mathcal{P}\Xi = \Psi$ by following the same argument as for Lemma 7.2. If $\mu_2 \neq 0$, then without loss of generality, we can assume that $\mu_2 = 1$, and the arguments presented below will be similar to the proof of Lemma 7.2.

Differentiating $(7.23)_1$ with respect to y' and $(7.23)_2$ with respect to x', and subtracting these two equations, we find

$$\Delta' \xi_2 = \psi_{1y'} - \psi_{2x'}. \tag{7.24}$$

We first associate with equation (7.24) the boundary condition

$$\xi_2 = 0, \text{ on } \Gamma_1', \tag{7.25}$$

which is already contained in ($7.19^\sharp$). For the other boundaries, suitable boundary conditions can be obtained as follows. As in Lemma 7.2, we denote by ν_i' the unit normal vector to Γ_i', and τ_i' the unit tangent vector on Γ_i', for all $i \in \{1,2,3,4\}$. On Γ_2', we have $\partial\xi_1/\partial\tau_2' = 0$ since $\xi_1 = 0$ on Γ_2'. Noticing that $(0,1)$ is parallel to τ_2', we find that $\xi_{1y'} = 0$, which, together with $(7.23)_2$, implies that $\xi_{2x'} = -\xi_{1y'} = 0$ by noticing that ψ_2 vanishes on Γ_2'. Observing that ν_2' is parallel to $(1,0)$, we then associate to (7.24) the following boundary condition on Γ_2':

$$\frac{\partial\xi_2}{\partial\nu_2'} = 0, \text{ on } \Gamma_2'. \tag{7.26}$$

Similar arguments show that we need to associate to (7.24) the following condition on the other two boundaries:

$$\begin{cases} \frac{\partial\xi_2}{\partial\nu_3'} + \frac{\partial\xi_2}{\partial\tau_3'} = 0, \text{ on } \Gamma_3', \\ \frac{\partial\xi_2}{\partial\nu_4'} - \frac{\partial\xi_2}{\partial\tau_4'} = 0, \text{ on } \Gamma_4'. \end{cases} \tag{7.27}$$

The existence of a (possibly non-unique) solution $\xi_2 \in H^1(\Omega')$ of (7.24)-(7.27) follows from Lemma 4.4.4.2 of [Gri85] with $\Omega = \Omega'$, $\mathscr{D} = \Gamma_1'$, $\mathscr{N} = \Gamma_2' \cup \Gamma_3' \cup \Gamma_4'$, $f = \psi_{1y'} - \psi_{2x'}$. Integrating $(7.24)_2$ with respect to y' and using the boundary condition $(7.19^\sharp)_3$, we obtain a solution ξ_1 which belongs to $H^1(\Omega')$.

In conclusion, in the new coordinate system (x', y'), we find a (possibly non-unique) solution Ξ that belongs to $\overline{V}^\sharp$, and solves the problem $\mathcal{P}\Xi = \Psi$. Transforming back to the original system (x, y) and the original variables Θ, we obtain $\overline{\Theta} \in \overline{V}$ satisfying $\mathcal{T}^*\overline{\Theta} = \Psi$. Therefore, the proof of Lemma 7.3 is complete. □

8. The mixed subcritical case

In this section, we consider the stationary 2D shallow water equations operator for the last case which includes both hyperbolic and elliptic modes. We assume that

$$u_0^2 + v_0^2 < g\phi_0, \tag{8.1}$$

which implies that

$$u_0^2 < g\phi_0, \; v_0^2 < g\phi_0. \tag{8.2}$$

Here, we choose the following boundary conditions according to the discussion in Subsection 2.4:

$$\begin{cases} v_0 u - u_0 v = u_0 u + v_0 v + g\phi = 0, \text{ on } \Gamma_1 \cup \Gamma_3 = \{x = 0\} \cup \{y = 0\}, \\ \phi = 0, \text{ on } \Gamma_2 \cup \Gamma_4 = \{x = L_1\} \cup \{y = L_2\}. \end{cases} \tag{8.3}$$

We then define the function space

$$\mathcal{D}(A) = \{U \in H = L^2(\Omega)^3, \mathcal{A}U \in H, \text{ and } U \text{ satisfies (8.3)}\},$$

and set $AU = \mathcal{A}U$, $\forall U \in \mathcal{D}(A)$.

8.1. **Positivity of A and its adjoint A^*.** We use the notations ξ, η, ζ, P, S_0 introduced in Subsection 2.4, and set $\Xi = (\xi, \eta, \zeta)^t$, so that $U = P\Xi$. We then write

$$\overline{A}_1 = P^t S_0 \mathcal{E}_1 P = \frac{1}{u_0^2 + v_0^2} \begin{pmatrix} u_0 & \frac{gv_0}{\kappa_1} & 0 \\ \frac{gv_0}{\kappa_1} & -u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix},$$

$$\overline{A}_2 = P^t S_0 \mathcal{E}_2 P = \frac{1}{u_0^2 + v_0^2} \begin{pmatrix} v_0 & -\frac{gu_0}{\kappa_1} & 0 \\ -\frac{gu_0}{\kappa_1} & -v_0 & 0 \\ 0 & 0 & v_0 \end{pmatrix},$$

and for $U \in \mathcal{D}(A)$, we compute:

$$\begin{aligned} \langle AU, U\rangle_H &= \int_\Omega (u_0 u_x + v_0 u_y + g\phi_x)u + (u_0 v_x + v_0 v_y + g\phi_y)v \\ &\quad + \frac{g}{\phi_0}(u_0\phi_x + v_0\phi_y + \phi_0(u_x + v_y))\phi dxdy \\ &= \langle S_0\mathcal{E}_1 U_x + S_0\mathcal{E}_2 U_y, U\rangle_H \\ &= \langle P^t S_0 \mathcal{E}_1 P \Xi_x + P^t S_0 \mathcal{E}_2 P \Xi_y, \Xi\rangle_H \\ &= \frac{1}{u_0^2+v_0^2}\int_\Omega (\xi,\eta)\begin{pmatrix} u_0 & \frac{gv_0}{\kappa_1} \\ \frac{gv_0}{\kappa_1} & -u_0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}_x + (\xi,\eta)\begin{pmatrix} v_0 & -\frac{gu_0}{\kappa_1} \\ -\frac{gu_0}{\kappa_1} & -v_0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}_y dxdy \\ &\quad + \frac{1}{u_0^2+v_0^2}\int_\Omega \zeta(u_0\zeta_x + v_0\zeta_y)dxdy. \end{aligned} \tag{8.4}$$

We now need to apply integration by parts for the two terms J_0, J_1 in the right-hand side of (8.4). According to Theorem 7.1, we obtain that ξ, η actually belong to $H^1(\Omega)$ since ξ, η satisfy the elliptic equations (2.17). Hence, integration by parts are valid for J_0, and we find

$$\begin{aligned} J_0 =& \frac{1}{2(u_0^2+v_0^2)}\int_0^{L_2} (\xi,\eta)\begin{pmatrix} u_0 & \frac{gv_0}{\kappa_1} \\ \frac{gv_0}{\kappa_1} & -u_0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\Big|_{x=0}^{x=L_1} dy \\ &+ \frac{1}{2(u_0^2+v_0^2)}\int_0^{L_1} (\xi,\eta)\begin{pmatrix} v_0 & -\frac{gu_0}{\kappa_1} \\ -\frac{gu_0}{\kappa_1} & -v_0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\Big|_{y=0}^{y=L_2} dx. \end{aligned}$$

For J_1, with density Theorem 3.1 applied with $\theta = \zeta, \lambda = u_0/v_0, \Gamma = \Gamma_1 \cup \Gamma_3$, we see that the integration by parts are also valid for J_1, and we find

$$J_1 = \frac{1}{2(u_0^2+v_0^2)}\Big(\int_0^{L_2} u_0\zeta^2\Big|_{x=0}^{x=L_1} dy + \int_0^{L_1} v_0\zeta^2\Big|_{y=0}^{y=L_2} dx\Big).$$

Combining the results for J_0 and J_1, we arrive at (written in a compact form):

$$\begin{aligned} \langle AU, U\rangle_H &= \frac{1}{2}\int_0^{L_2} \Xi^t\overline{A}_1\Xi\Big|_{x=0}^{x=L_1} dy + \frac{1}{2}\int_0^{L_1} \Xi^t\overline{A}_2\Xi\Big|_{y=0}^{y=L_2} dx \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{8.5}$$

where I_i stands for the boundary term at Γ_i for $i = 1, 2, 3, 4$. Using the boundary conditions (8.3), we now estimate $I_i, (i = 1, 2, 3, 4)$ as follows:

$$
\begin{aligned}
2(u_0^2 + v_0^2)I_1 &= -(u_0\xi^2 + 2\frac{gv_0}{\kappa_1}\xi\eta - u_0\eta^2 + u_0\zeta^2)(0, y) = (u_0\eta^2)(0, y) \geq 0,\\
2(u_0^2 + v_0^2)I_2 &= (u_0\xi^2 + 2\frac{gv_0}{\kappa_1}\xi\eta - u_0\eta^2 + u_0\zeta^2)(L_1, y) = (u_0\xi^2 + u_0\zeta^2)(0, y) \geq 0,\\
2(u_0^2 + v_0^2)I_3 &= -(v_0\xi^2 - 2\frac{gu_0}{\kappa_1}\xi\eta - v_0\eta^2 + v_0\zeta^2)(x, 0) = (v_0\eta^2)(x, 0) \geq 0,\\
2(u_0^2 + v_0^2)I_4 &= (v_0\xi^2 - 2\frac{gu_0}{\kappa_1}\xi\eta - v_0\eta^2 + v_0\zeta^2)(x, L_2) = (v_0\xi^2 + v_0\zeta^2)(x, L_2) \geq 0,
\end{aligned}
$$

which are all nonnegative. Therefore, we conclude that $\langle AU, U\rangle_H \geq 0$ for all $U \in \mathcal{D}(A)$. Note that here, because the part of the operator A which is elliptic gives some additional regularity, we did not need to use an approximation argument to show that $\langle AU, U\rangle_H \geq 0$, $\forall U \in \mathcal{D}(A)$, unlike in the previous cases.

We now turn to the adjoint A^*. In the mixed subcritical case, the formal definition of A^* can also be treated similarly as in Subsection 4.1, we thus omit the details. The adjoint differential operator $\mathcal{A}^*$ is given by (4.7) again. In order to guarantee that $U \mapsto \langle AU, \overline{U}\rangle_H$ is continuous on $\mathcal{D}(A)$, the following boundary conditions in the (ξ, η, ζ) variables need to be satisfied (see (7.19) since the equations for ξ, η are elliptic):

$$
\begin{cases}
\frac{gv_0}{\kappa_1}\bar{\xi} - u_0\bar{\eta} = 0, \text{ on } \Gamma_1 = \{x = 0\},\\
\bar{\zeta} = u_0\bar{\xi} + \frac{gv_0}{\kappa_1}\bar{\eta} = 0, \text{ on } \Gamma_2 = \{x = L_1\},\\
-\frac{gu_0}{\kappa_1}\bar{\xi} - v_0\bar{\eta} = 0, \text{ on } \Gamma_3 = \{y = 0\},\\
\bar{\zeta} = v_0\bar{\xi} - \frac{gu_0}{\kappa_1}\bar{\eta} = 0, \text{ on } \Gamma_4 = \{y = L_2\}.
\end{cases}
\tag{8.6}
$$

Transforming back to the (u, v, ϕ) variables, the boundary conditions read

$$
\begin{cases}
gv_0^2\bar{u} - gv_0u_0\bar{v} - u_0\kappa_1^2\bar{\phi} = 0, \text{ on } \Gamma_1 = \{x = 0\},\\
u_0v_0\bar{u} - u_0^2\bar{v} + gv_0\bar{\phi} = u_0\bar{u} + v_0\bar{v} + g\bar{\phi} = 0, \text{ on } \Gamma_2 = \{x = L_1\},\\
gu_0v_0\bar{u} - gu_0^2\bar{v} + v_0\kappa_1^2\bar{\phi} = 0, \text{ on } \Gamma_3 = \{y = 0\},\\
v_0^2\bar{u} - v_0u_0\bar{v} - gu_0\bar{\phi} = u_0\bar{u} + v_0\bar{v} + g\bar{\phi} = 0, \text{ on } \Gamma_4 = \{y = L_2\},
\end{cases}
\tag{8.7}
$$

which are more complicated than the boundary conditions (8.3), since some elliptic modes are hidden in the operator A. We conclude that

$$
\mathcal{D}(A^*) = \{\overline{U} \in H = L^2(\Omega)^3, A^*\overline{U} \in H, \text{ and } \overline{U} \text{ satisfies (8.7)}\},
$$

and set $A^*\overline{U} = \mathcal{A}^*\overline{U}$, $\forall\overline{U} \in \mathcal{D}(A^*)$.

The proof of the positivity of A^* is similar to that of A, we thus omit it here.

9. The initial and boundary value problem for the full system

In this section we aim to combine the results of the previous sections and to investigate the well-posedness for Eqs. (1.1) associated with the suitable initial and boundary conditions that we have introduced. Here again, we consider the case of homogeneous boundary conditions (see below a remark about the non-homogeneous boundary conditions).

As we have already seen in previous sections, we have five cases to consider depending on the relations between the constants u_0, v_0, ϕ_0, g. In the following, we list these relations and their corresponding boundary conditions:

The supercritical case. The assumption is

$$u_0^2 > g\phi_0, \;\; v_0^2 > g\phi_0, \tag{9.1}$$

and the corresponding boundary conditions are

$$\begin{cases} u = v = \phi = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ u = v = \phi = 0, \text{ on } \Gamma_3 = \{y = 0\}; \end{cases} \tag{9.1$'$}$$

The mixed case I. The assumption is

$$u_0^2 < g\phi_0, \;\; v_0^2 > g\phi_0, \tag{9.2}$$

and the corresponding boundary conditions are

$$\begin{cases} v_0 u - u_0 v + \kappa_0 \phi = u_0 u + v_0 v + g\phi = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ v_0 u - u_0 v - \kappa_0 \phi = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ u = v = \phi = 0, \text{ on } \Gamma_3 = \{y = 0\}; \end{cases} \tag{9.2$'$}$$

The mixed case II. The assumption is

$$u_0^2 > g\phi_0, \;\; v_0^2 < g\phi_0, \tag{9.3}$$

and the corresponding boundary conditions are

$$\begin{cases} v_0 u - u_0 v - \kappa_0 \phi = u_0 u + v_0 v + g\phi = 0, \text{ on } \Gamma_3 = \{y = 0\}, \\ v_0 u - u_0 v + \kappa_0 \phi = 0, \text{ on } \Gamma_4 = \{y = L_2\}, \\ u = v = \phi = 0, \text{ on } \Gamma_1 = \{x = 0\}; \end{cases} \tag{9.3$'$}$$

The fully hyperbolic subcritical case. The assumption is

$$u_0^2 < g\phi_0, \;\; v_0^2 < g\phi_0, \;\; u_0^2 + v_0^2 > g\phi_0, \tag{9.4}$$

and the corresponding boundary conditions are

$$\begin{cases} v_0 u - u_0 v + \kappa_0 \phi = u_0 u + v_0 v + g\phi = 0, \text{ on } \Gamma_1 = \{x = 0\}, \\ v_0 u - u_0 v - \kappa_0 \phi = 0, \text{ on } \Gamma_2 = \{x = L_1\}, \\ v_0 u - u_0 v - \kappa_0 \phi = u_0 u + v_0 v + g\phi = 0, \text{ on } \Gamma_3 = \{y = 0\}, \\ v_0 u - u_0 v + \kappa_0 \phi = 0, \text{ on } \Gamma_4 = \{y = L_2\}; \end{cases} \tag{9.4$'$}$$

The mixed subcritical case. The assumption is

$$u_0^2 < g\phi_0, \;\; v_0^2 < g\phi_0, \;\; u_0^2 + v_0^2 < g\phi_0, \tag{9.5}$$

and the corresponding boundary conditions are

$$\begin{cases} v_0 u - u_0 v = u_0 u + v_0 v + g\phi = 0, \text{ on } \Gamma_1 \cup \Gamma_3 = \{x = 0\} \cup \{y = 0\}, \\ \phi = 0, \text{ on } \Gamma_2 \cup \Gamma_4 = \{x = L_1\} \cup \{y = L_2\}. \end{cases} \tag{9.5$'$}$$

If these constants u_0, v_0, ϕ_0, g satisfy assumption (9.i), then we define the domain of the unbounded operator A:

$$\mathcal{D}(A) = \{U \in H = L^2(\Omega)^3, AU \in H, \text{ and } U \text{ satisfies } (9.i')\},$$

where $i \in \{1, 2, 3, 4, 5\}$, indicating the five different cases.

We set $BU = (-fv, fu, 0)^t$, where f is the Coriolis parameter, and it is easy to see that B is a linear continuous self-adjoint operator on H. We also set $A_0 = A + B$, with $\mathcal{D}(A_0) = \mathcal{D}(A)$, and the adjoint $A_0^* = A^* + B^*$, with $\mathcal{D}(A_0^*) = \mathcal{D}(A^*)$. Then we have

Theorem 9.1. *The operator $-A_0$ is the infinitesimal generator of a contraction semigroup on H.*

Proof. According to [Yos80, HP74], it suffices to show that

(i) A_0 and A_0^* are both closed operators, and their domains $\mathcal{D}(A_0)$ and $\mathcal{D}(A_0^*)$ are both dense in H.
(ii) A_0 and A_0^* are both positive.

Noticing that B is a linear continuous self-adjoint operator on H and that $\langle BU, U\rangle = 0$ for all $U \in H$, we find that proving (i) and (ii) is equivalent to proving the following:

(i′) A and A^* are both closed operators, and their domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are both dense in H.
(ii′) A and A^* are both positive.

We already showed that A and A^* are both positive in the previous subsections, we thus only need to prove (i′). We establish the result for A, and the proof for A^* would be similar.

In all cases, observing that $\mathcal{D}(\Omega)^3$ is included in $\mathcal{D}(A)$ and dense in $H = L^2(\Omega)^3$, we thus obtain $\mathcal{D}(A)$ is dense in H. To show that A is closed, consider a sequence $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ for which $\lim_{n\to\infty} U_n = U$ in $L^2(\Omega)^3$ and $\lim_{n\to\infty} AU_n = \widehat{U}$ in $L^2(\Omega)^3$. By the L^2-convergence of $\{U_n\}_{n\in\mathbb{N}}$, we first find that AU_n converges to AU in the sense of distributions, which implies that AU equals $\widehat{U}$ in the sense of distributions. Since $\widehat{U}$ belongs to $L^2(\Omega)^3$, we obtain that AU belongs to $L^2(\Omega)^3$, too. Thus, we have $U, AU \in L^2(\Omega)^3$, which shows that the traces of U are well-defined thanks to Proposition 2.1. By Proposition 2.1 again, the traces of U_n converge weakly to the traces of U in the appropriate space H^{-1}, so that U satisfies the boundary conditions (9.i'), where $i \in \{1, 2, 3, 4, 5\}$. Therefore, we conclude that $U \in \mathcal{D}(A)$. Hence A is closed, and the proof is complete. □

We now consider the whole system of 2D linearized Shallow Water Equations, namely (1.1) and introduce the initial and boundary conditions. We consider the homogeneous boundary conditions and will briefly explain how to treat the case of non-homogeneous boundary conditions.

As we discussed before, the boundary conditions are (9.i'), $i \in \{1, 2, 3, 4, 5\}$ depending on the case we are considering, and all these boundary conditions are taken into account in the domain $\mathcal{D}(A_0)$ of A_0. Finally if we add the initial conditions:

$$U(0) = (u(0), v(0), \phi(0)) = U_0 = (u_0, v_0, \phi_0), \tag{9.6}$$

then the initial and boundary value problem consisting of Eqs. (1.1), $(9.i'), i \in \{1,2,3,4,5\}$ and (9.6) is equivalent to the abstract initial value problem

$$\begin{cases} \frac{dU}{dt} + A_0 U = F, \\ U(0) = U_0. \end{cases} \tag{9.7}$$

Note that $F = (F_u, F_v, F_\phi)$ which does not appear in (1.1) is added here for mathematical generality and to study the case of non-homogeneous boundary conditions. Thanks to Theorem 9.1 this problem is now solved by the Hille-Yoshida theorem and we have that:

Theorem 9.2. *Let H, A_0 and $\mathcal{D}(A_0)$ be defined as before. Then the initial value problem (9.7) is well-posed. That is,*

i) for every $U_0 \in H$, and $F \in L^1(0,T;H)$, the problem (9.7) admits a unique weak solution $U \in \mathcal{C}([0,T];H)$ satisfying

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds, \; \forall t \in [0,T],$$

where $(S(t))_{t\geq 0}$ is the contraction semigroup generated by the operator $-A_0$;

ii) for every $U_0 \in \mathcal{D}(A_0)$, and $F \in L^1(0,T;H)$, with $F' = dF/dt \in L^1(0,T;H)$, the problem (9.7) has a unique strong solution U such that

$$U \in \mathcal{C}\big([0,T];H\big) \cap L^\infty\big(0,T;\mathcal{D}(A_0)\big), \;\; \frac{dU}{dt} \in L^\infty\big(0,T;H\big).$$

Remark 9.1 (Non-homogeneous boundary conditions)**.** *In the case of non-homogeneous boundary conditions, that is if we want to solve (1.1), with $(9.i')$, $i \in \{1,2,3,4,5\}$, in which the boundary conditions are now non-homogeneous, and with initial condition (9.6), we classically proceed by translation. We assume that the boundary data are inferred from a function Φ which is defined on $\Omega \times [0,T]$, and we set $V = U - \Phi$. The problem is now to check the hypotheses of Theorem 9.2 for the data corresponding to V and to apply Theorem 9.2. We omit the details; see* [RTT08b] *for a similar situation.*

Remark 9.2. *The linearized 3D inviscid Primitive Equations of the atmosphere and the oceans can be written:*

$$\begin{cases} u_t + U_0 u_x + V_0 u_y - fv + \phi_x = 0, \\ v_t + U_0 v_x + V_0 v_y + fu + \phi_y = 0, \\ T_t + U_0 T_x + V_0 T_y + N^2 \frac{T_0}{g} w = 0, \\ u_x + v_y + w_z = 0, \\ \phi_z = \frac{gT}{T_0}, \end{cases} \tag{9.8}$$

where (u,v,w) is the velocity of the water, (u,v) the horizontal velocity, T the temperature, ϕ a multiple of the pressure; g the gravitational acceleration, and N^2 denotes the Brunt-Väisälä (buoyancy) frequency satisfying

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz},$$

and $U_0 > 0, V_0 > 0, \rho_0 > 0$ and $T_0 > 0$ are reference average values of the density and the temperature.

We refer to [RTT08b] *for a systematic discussion in the case when* $V_0 = 0$*. Using the normal modes expansion for (9.8) (see details in* [RTT08b]*), we obtain the equations for the non-zero modes* $(u_n, v_n, \phi_n, \lambda_n)$ *which read (the indices* n *are dropped for the sake of simplicity):*

$$\begin{cases} u_t + U_0 u_x + V_0 u_y - fv - \frac{1}{\lambda}\psi_x = 0, \\ v_t + U_0 v_x + V_0 v_y + fu - \frac{1}{\lambda}\psi_y = 0, \\ \psi_t + U_0 \psi_x + V_0 \psi_y - \frac{N^2}{\lambda}(u_x + v_y) = 0, \end{cases} \tag{9.9}$$

where $\psi = \phi_z$ *and* λ *is positive. Observing that (9.9) has the same form as the shallow water equation (1.1) if we replace* ϕ *and* g *in (1.1) by* $-\psi$ *and* λ^{-1}*, we thus can obtain a well-posedness result (see Theorem 9.2) for (9.9). The equations for the zero mode are*

$$\begin{cases} u_t + U_0 u_x + V_0 u_y - fv + \frac{1}{\lambda}\phi_x = 0, \\ v_t + U_0 v_x + V_0 v_y + fu + \frac{1}{\lambda}\phi_y = 0, \\ u_x + v_y = 0. \end{cases} \tag{9.10}$$

Following the same argument as for the zero mode in [RTT08b] *and* [CST10]*, we can also obtain the well-posedness result for (9.10). Therefore, we can obtain a well-posedness result for the whole system (9.8) with suitable boundary conditions and initial conditions as in Section 4 of* [RTT08b]*. The details will appear elsewhere.*

Acknowledgments

This work was partially supported by the National Science Foundation under the grants NSF DMS-0906440 and DMS-1206438, and by the Research Fund of Indiana University.

References

[AB03] E. Audusse and M.-O. Bristeau, *Transport of pollutant in shallow water: A two time steps kinetic method*, M2AN Math. Model. Numer. Anal. **37** (2003), no. 2, 389–416.

[ABBKP] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, and B. Perthame, *A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows*, SIAM Journal of Scientic Computing **25** (2004), no. 6, 2050–2065.

[ABPS] E. Audusse, M.O. Bristeau, B. Perthame, and J. Sainte-Marie, *A multilayer saint-venant system with mass exchanges for shallow water flows. derivation and numerical validation*, M2AN Math. Model. Numer. Anal. **45 (1)** (2011), 169–200.

[BS07] S. Benzoni-Gavage and D. Serre, *Multi-dimensional Hyperbolic Partial Differential Equations*, Oxford University Press, 2007.

[BPSTT] A. Bousquet, M. Petcu, M.-C. Shiue, R. Temam, and J. Tribbia, *Boundary conditions for limitied area models*, To appear.

[CST10] Q. Chen, M.-C. Shiue, and R. Temam, *The barotropic mode for the primitive equations, Speical issue in momory of David Gottlieb*, Journal of Scientific Computing **45** (2010), 167–199.

[Eva98] L. C. Evans, *Partial Differential Equations*, vol. 19, Amer. Math. Soc., Providence, RI, 1998.

[Gri85] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, Pitman, Boston, 1985.

[Hor65] L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), 89–152.

[HPT11] A. Huang, M. Petcu, and R. Temam, *The one-dimensional supercritical shallow-water equations with topography*, Annals of the University of Bucharest (Mathematical Series) **2 (LX)** (2011), 63–82.

[HP74] E. Hille and R.S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, Providence, RI, 1974 (Third printing of the revised edition of 1957, AMS Colloquium Publications, vol. XXXI).
[Kre70] H.-O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math **23** (1970), 277–298.
[KT80] K. Kojima and M. Taniguchi, *Mixed problem for hyperbolic equations in a domain with a corner*, Funkcialaj Ekvacioj **23** (1980), 171–195.
[Lop70] Ya. B. Lopatinskii, *The mixed Cauchy-Dirichlet type problem for equations of hyperbolic type*, Dopovfdf Akad. Nauk Ukrai"n. RSR Ser. A **668** (1970), 592–594.
[OS78] J. Oliger and A. Sundström, *Theoretical and practical aspects of some initial-boundary value problems in fluid dynamics*, SIAM J. Appl. Math. **35 (3)** (1978), 419–446.
[Ore95] P. Orenga, *Un théorème d'existence de solutions d'un probléme de shallow water*, Archive for Rational Mechanics and Analysis **130** (1995), 183–204.
[Osh73] Stanley Osher, *Initial-boundary value problems for hyperbolic systems in regions with corners. I*, Trans. Amer. Math. Soc. **176** (1973), 141–165.
[Osh74] ______, *Initial-boundary value problems for hyperbolic systems in regions with corners. II*, Trans. Amer. Math. Soc. **198** (1974), 155–175.
[PT10] M. Petcu and R. Temam, *The one dimensional shallow water equations with Dirichlet boundary conditions on the velocity*, Discrete and Continuous Dynamical Systems Series S **4** (2010), no. 1, 209–222.
[PT11] ______, *The one-dimensional shallow water equations with transparent boundary conditions*, Math. Meth. Appl. Sci. (2011), DOI: 10.1002/mma.1482.
[RTT08a] A. Rousseau, R. Temam, and J. Tribbia, *Boundary value problems for the inviscid primitive equations in limited domains*, in *Computational Methods for the Atmosphere and the Oceans*, Special Volume of the *Handbook of Numerical Analysis*, vol. XIV, Roger M. Temam and Joseph J. Tribbia, Guest Editors, P.G. Giarlet Editor, Elsevier, Amsterdam, 2008.
[RTT08b] ______, *The 3D Primitive Equations in the absence of viscosity: Boundary conditions and well-posedness in the linearized case*, J. Math. Pures Appl. **89** (2008), 297–319.
[Rud91] W. Rudin, *Functional Analysis*, Second ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.
[SLTT] M.-C. Shiue, J. Laminie, R. Temam, and J. Tribbia, *Boundary value problems for the shallow water equations with topography*, Journal of Geophysical Research-Oceans **116** (2011), C02015.
[Tan78] Masaru Taniguchi, *Mixed problem for wave equation in the domain with a corner*, Funkcialaj Ekvacioj **21** (1978), 249–259.
[TT03] R. Temam and J. Tribbia, *Open boundary conditions for the primitive and Boussinesq equations*, J. Atmospheric Sci. **60 (21)** (2003), 2647–2660.
[WPT97] T. Warner, R. Peterson, and R. Treadon, *A tutorial on lateral boundary conditions as a basic and potentially serious limitation to regional numerical weather prediction*, Bull. Amer. Meteor. Soc. (1997), 2599–2617.
[Yos80] K. Yosida, *Functional Analysis*, Sixth ed., Springer-Verlag, Berlin, 1980.

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